

About Bounded Transformations of the Gamma Degradation Process

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Although the degradation processes of technological units are naturally bounded, due to the finiteness of their physical dimensions and/or the nature itself of the degradation mechanism, the models adopted to describe degradation phenomena are typically unbounded. In general, this apparent contradiction does not significantly affect the effectiveness of unbounded degradation models, because degrading units are conventionally considered failed when their degradation level exceeds a threshold value that is quite far from the “natural” bounds. On the other side, however, the effectiveness of an unbounded degradation models can drastically diminish if the physical bound and threshold have comparable values. The aim of this paper is then to investigate the potentiality of the transformed gamma process in modelling bounded degradation phenomena. This idea is not new. Yet, differently than in other existing models, here the upper bound is treated as an unknown parameter and is estimated from the available degradation data. The proposed approach, which led to the definition of a bounded (state-dependent) transformed gamma process, is illustrated starting with a motivating example, which is developed on the basis of a real set of wear data of cylinder liners equipping a diesel engine for marine propulsion. Model parameters are estimated by using the maximum likelihood method. Fitting ability of the innovative proposed bounded process is compared with those of the unbounded gamma process, previously adopted to analyze these wear data. Potentiality of the proposed approach are critically discussed in the paper.

Keywords: Bounded degradation phenomena, transformed gamma process, remaining useful life, residual reliability, maximum likelihood estimation.

1. Introduction

The degradation processes of technological units are usually described by using stochastic models, such as the gamma, the inverse Gaussian, and the Wiener ones, which assume that the degradation level can increase indeterminately. In the practice, this assumption is not realistic, if only for the fact that technological units have finite dimensions. For example, it is obvious that an upper bound surely exists in the cases of monotonic increasing degradation processes that describe the percentage reduction of quality of a unit with respect to its initial value, which (by definition) is constrained to range between 0 and 100. Again, it is equally obvious that an upper bound also exists when the degradation level of a unit is quantified in terms of loss of material. Nonetheless, the existence of these “obvious” bounds does not seriously affect the performance of unbounded degradation models when, as generally occurs, degrading units are considered failed when their degradation level passes a threshold that is typically largely far from them.

On the other side, in several cases the degradation level cannot reach the aforementioned “obvious” bounds due to inherent features of the mechanism that causes the degradation. Unfortunately, in most of these cases, even if it is easy to recognize that a bound of this kind exists, it is not simple to determine a priori its exact value, because

the degradation causing mechanism is not sufficiently well understood. For example, this latter situation could occur when the degradation is caused by friction between moving parts.

Since neglecting the presence of an upper bound, when it exists, could lead to poor estimates and/or wrong predictions, we believe that it is important to assess whether, and in which extent, the use of degradation models that explicitly account for the presence of an upper bound could be beneficial.

At this aim, and focusing our interest on increasing degradation phenomena, in this paper we investigate the potentiality of some bounded characterizations of the transformed gamma process (Giorgio et al. (2105a)) to tackle the aforementioned problem. Indeed, the transformed gamma process appears to be a good candidate, because it assumes that the degradation increments are not independent and the bounded degradation process obviously cannot have independent increments (for example, because its variance should necessarily go to zero as the degradation level approaches the upper bound).

We note that this idea is not new. In fact, just Giorgio et al. (2105a) used a bounded transformed gamma process with a log-based state function to describe the percentage reduction of strength of certain polymers. Contemporarily,

Ling et al. (2015) used a gamma process to model the opposite of the logarithm of the percentage reduction of light intensity of some light emitting diode (LED) lamps. Both these models are formulated assuming that the upper bound is known a priori, relying on the simple consideration that the reduction of the considered quality index could not exceed the initial value. Although the motivating ideas of these papers are slightly different, the proposed modelling solutions are very similar. A third bounded transformation of the gamma process is proposed by Deng and Pandey (2017), who used a logit transformation of the gamma process to model the wall thickness reduction of certain pipes. Once again, the upper bound of the degradation process is assumed known a priori and is set equal to the initial wall thickness.

Differently than in these papers, here we assume that the upper bound is one of the unknown parameters of the transformed gamma process. Hence, according to our settings, it has to be estimated from the available degradation data. Both to motivate the proposed modelling solution and to investigate its potentiality, we apply and compare some models obtained under these settings to a set of wear measurements of the liners of a 8-cylinder diesel engine equipping a cargo ship of the Grimaldi Lines.

The specific interest for the considered degradation phenomenon is motivated by the coexistence of three circumstances. The first one is that a liner is assumed to fail when its wear level exceeds 4mm, a value that is hugely smaller than the thickness of the liner itself, which is equal to 100 mm. The second reason is that the observed wear mechanism cannot lead the liner wear to lead up its thickness. In fact, the upper bound is expected to be not too larger than the threshold limit. The third reason is that the available prior knowledge does not allow to determine the exact value of the upper bound of the process, which (consequently) has to be estimated from the observed data.

Note that, under these circumstances, assuming that the upper bound coincides with the thickness of the liner is practically equivalent to assume that the degradation process is unbounded.

The rest of the paper is structured as it follows. The bounded transformed gamma process is introduced in Section 2, while the expressions of the corresponding reliability function and conditional distribution of the remaining useful life are given in Section 3. The maximum likelihood estimators of model parameters are formulated in Section 4, and an applicative example referring to the wear data of the cylinder liners of the diesel engine is developed in Section 5. Final considerations are given in Section 6.

2. The Bounded Transformed Gamma Process

Let $\eta(t)$ be a non-negative, monotone increasing function of time t , with $\eta(0) = 0$, that will be called hereinafter as the age function. The increasing bounded degradation process $\{W(t); t \geq 0\}$ is assumed to be a bounded transformed gamma (BTG) process if:

- (i) the degradation increments over disjoint time intervals are not independent,

- (ii) the degradation increment $\Delta W(t, t + \Delta t) \equiv W(t + \Delta t) - W(t)$ over the time interval $(t, t + \Delta t)$ depends on the process history up to t through the current time t and the current degradation level (status) $w_t = W(t)$, being independent on the past,

- (iii) the conditional probability density function (pdf) of $\Delta W(t, t + \Delta t)$ is given by:

$$f_{\Delta W(t, t + \Delta t)}(\delta | w_t) = g'(w_t + \delta) \times \frac{[g(w_t, w_t + \delta)]^{\eta(t, t + \Delta t) - 1} \exp[-g(w_t, w_t + \delta)]}{\Gamma[\eta(t, t + \Delta t)]}, \quad 0 < \delta < w_{lim} - w_t, \quad (1)$$

where $g(w)$ is a non-negative, monotone increasing and differentiable function of the degradation level w ($0 < w < w_{lim}$), called the state function, with $g(0) = 0$ and

$$\lim_{w \rightarrow w_{lim}} g(w) = \infty, \quad (2)$$

$g'(w_t + \delta)$ is its first derivative evaluated at $w_t + \delta$, $g(w_t, w_t + \delta) = g(w_t + \delta) - g(w_t)$, $\eta(t, t + \Delta t) = \eta(t + \Delta t) - \eta(t)$, and $\Gamma[\eta(t, t + \Delta t)]$ is the complete gamma function.

Of course, the BTG process is fully defined once the state and the age functions are defined. Suitable forms for the “bounded” state function are:

$$g_1(w) = -\beta \ln(1 - w/w_{lim}) \quad (3)$$

$$g_2(w) = \beta \frac{w}{w_{lim} - w} \quad (4)$$

$$g_3(w) = \beta \tan\left(\frac{\pi}{2} \frac{w}{w_{lim}}\right), \quad (5)$$

whose first derivatives are, respectively:

$$g'_1(w) = \frac{\beta}{w_{lim} - w} \quad (6)$$

$$g'_2(w) = \frac{\beta w_{lim}}{(w_{lim} - w)^2} \quad (7)$$

$$g'_3(w) = \frac{\beta}{w_{lim}} \frac{\pi/2}{\cos^2\left(\frac{\pi}{2} \frac{w}{w_{lim}}\right)}. \quad (8)$$

Suitable forms for the age function are those already proposed for the TG process, e.g. $\eta(t) = (t/a)^b$ and $\eta(t) = ab[\exp(t/b) - 1]$ (Giorgio et al. (2015a) and (2015b)). Note that when $\eta(t)$ is linear with the age t , so that $\eta(t, t + \Delta t) \propto \Delta t$, then from (1) we have that the conditional distribution of the degradation increment $\Delta W(t, t + \Delta t)$, given the current state w_t , does not depend on the current age t . Otherwise, the conditional distribution of $\Delta W(t, t + \Delta t)$, given the current state w_t , depends also on the current age t .

Similarly, the functional form of the state function $g(w)$ determines how, given t , the increment $\Delta W(t, t + \Delta t)$ depends on the current degradation state $W(t)$.

From (1), the conditional cumulative distribution function (Cdf) of the degradation increment $\Delta W(t, t + \Delta t)$ is given by:

$$F_{\Delta W(t,t+\Delta t)}(\delta|w_t) = \begin{cases} \frac{IG[g(w_t, w_t + \delta); \eta(t, t + \Delta t)]}{\Gamma[\eta(t, t + \Delta t)]}, & \text{for } \delta < w_{lim} - w_t \\ 1, & \text{for } \delta \geq w_{lim} - w_t \end{cases} \quad (9)$$

where:

$$IG(x; a) = \int_0^x u^{a-1} \exp(-u) du \quad (10)$$

is the (lower) incomplete gamma function. From (1) and (10), the pdf and the Cdf of the degradation level $W(t)$ at the time t of a new (unused) unit (that is, of a unit with age $t = 0$ and state $W(0) = 0$) are given by:

$$f_{W(t)}(w) = g'(w) \frac{[g(w)]^{\eta(t)-1}}{\Gamma[\eta(t)]} \exp[-g(w)], \quad 0 < w < w_{lim} \quad (11)$$

$$F_{W(t)}(w) = \begin{cases} \frac{IG[g(w); \eta(t)]}{\Gamma[\eta(t)]}, & \text{for } w < w_{lim} \\ 1, & \text{for } w \geq w_{lim} \end{cases} \quad (12)$$

The conditional mean and variance of the degradation increment, given the state w_t at the time t , are not in a closed form, and require numerical integration:

$$E\{\Delta W(t, t + \Delta t)|w_t\} = \int_0^{w_{lim}-w_t} \delta f_{\Delta W(t,t+\Delta t)}(\delta|w_t) d\delta \quad (13)$$

$$V\{\Delta W(t, t + \Delta t)|w_t\} = \int_0^{w_{lim}-w_t} \delta^2 f_{\Delta W(t,t+\Delta t)}(\delta|w_t) d\delta - E^2\{\Delta W(t, t + \Delta t)|w_t\}. \quad (14)$$

Likewise, the mean and variance of the degradation level $W(t)$ are given by:

$$E\{W(t)\} = \int_0^{w_{lim}} w f_{W(t)}(w) dw \quad (15)$$

$$V\{W(t)\} = \int_0^{w_{lim}} w^2 f_{W(t)}(w) dw - E^2\{W(t)\}. \quad (16)$$

Due to the “bounded” nature of the state function $g(w)$, for $t \rightarrow \infty$ the pdf (11) tends to the Dirac delta distribution with support w_{lim} , and hence the mean $E\{W(t)\}$ of the degradation process tends to w_{lim} for $t \rightarrow \infty$. In turns, the variance $V\{W(t)\}$, that initially grows with time, approaches to zero for $t \rightarrow \infty$.

Similarly, for $\Delta t \rightarrow \infty$, the conditional pdf (1) tends to the Dirac delta distribution with support $w_{lim} - w_t$, and hence the conditional mean $E\{\Delta W(t, t + \Delta t)|w_t\}$ of the degradation increment tends to $w_{lim} - w_t$, and the conditional variance $V\{\Delta W(t, t + \Delta t)|w_t\}$ approaches to zero for $\Delta t \rightarrow \infty$.

In addition, from (11), we have that, under the proposed state functions (3)-(5), the quantity w_{lim} acts as a scale parameter for the pdf $f_{W(t)}(w)$ of the degradation level at the time t , which then can be rewritten as:

$$f_{W(t)}(w) = \frac{1}{w_{lim}} h(z; \eta(t), \beta), \quad (17)$$

where $z = w/w_{lim}$ and the function $h(z; \eta(t), \beta)$, whose expression depends on the functional form of $\eta(t)$ and $g(w)$, is independent of w_{lim} . For example, by assuming the state function $g_1(w)$ in (3), we have that:

$$h(z; \eta(t), \beta) = \frac{\beta}{1-z} \frac{[-\beta \ln(1-z)]^{\eta(t)-1}}{\Gamma[\eta(t)]} (1-z)^\beta \quad 0 < z < 1.$$

This implies that, under the proposed state functions (3)-(5), the mean and the variance of $W(t)$ depend linearly on w_{lim} and w_{lim}^2 , respectively.

By assuming the state function $g_1(w)$ in (3) and the power-law age function $\eta(t) = (t/a)^b$, the curves of the degradation mean $E\{W(t)\}$ are depicted in Figure 1, for $w_{lim} = 10$ and different selected values of the process parameters a, b , and β , that is, (1, 1, 1), (1, 0.5, 1), (1, 1, 2), (1, 2, 5), and (3, 1, 1), respectively. We have that the mean curve has an inflection point when the age parameter b is larger than 1, that is, when the age function is convex.

In Figure 2, the variance $V\{W(t)\}$ of the same degradation model is depicted for $w_{lim} = 10$ and the same selected values of the process parameters a, b , and β used for Figure 1. Empirical studies showed that the maximum value $V_{max}\{W(t)\}$ of the variance does not depend on the values of the age parameters a and b , and hence the ratio $V_{max}\{W(t)\}/w_{lim}^2$ does not depend on a and b , but only on the parameter β of the state function.

In addition, given a, b , and α , the time at which the variance reaches its maximum value does not depend on the value of w_{lim} , and, given b and β , is proportional to a , irrespectively from the value of w_{lim} .

Similar behavior of the mean and variance curves and the same properties of the mean and variance of $W(t)$ are provided by assuming the state functions (4) and (5).

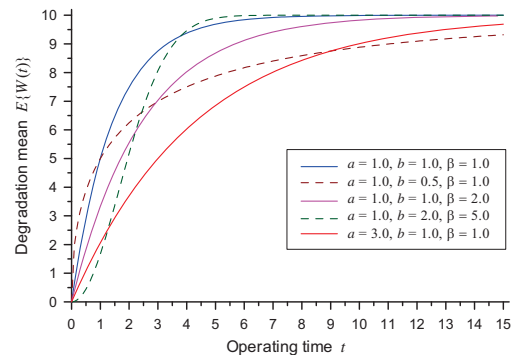


Fig. 1. Behavior of the degradation mean $E\{W(t)\}$, for $w_{lim} = 10$ and selected values of the process parameters a, b , and β .

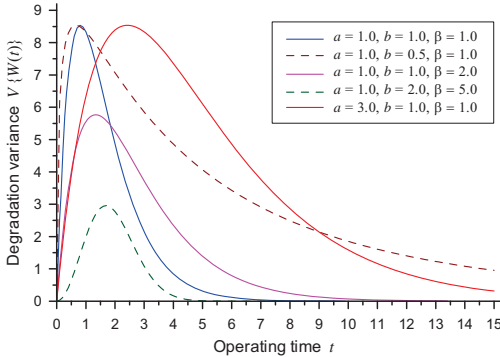


Fig. 2. Behavior of the degradation variance $V[W(t)]$, for $w_{lim} = 10$ and selected values of the process parameters a , b , and β .

3. The Remaining Useful Life and Reliability Function

In the context of increasing degradation processes, a unit is assumed to fail when its degradation level W exceeds a threshold limit D ($D < w_{lim}$). Then, the unit lifetime X can be defined as the operating time to first, and sole, passage beyond the limit D . The remaining useful life (RUL) X_t of a unit which is unfailed at the operating time t , say $X_t = X - t$, is defined as the further operating time the unit will spend to exceed the level D , when starting from the current degradation level $W(t) = w_t$.

Then, by using the conditional Cdf (9), the residual reliability, that is the conditional probability of the RUL X_t exceeding the time x , given the current state $w_t < D$ at the current age t , is given by:

$$R_t(x|w_t) = \Pr\{\Delta W(t, t+x) \leq D - w_t | w_t\} = \frac{IG[g(w_t, D); \eta(t, t+x)]}{\Gamma[\eta(t, t+x)]}. \quad (18)$$

From (18), the reliability function of a new unit is given by:

$$R(x) = \Pr\{W(x) \leq D\} = \frac{IG[g(D); \eta(x)]}{\Gamma[\eta(x)]}. \quad (19)$$

If the age function $\eta(t)$ is differentiable with respect to t , then the conditional pdf of the RUL X_t can be obtained by deriving (18) with respect to x :

$$f_{X_t}(x|w_t) = -\frac{d}{dx} \int_0^{g(w_t, D)} \frac{u^{\eta(t, t+x)-1}}{\Gamma[\eta(t, t+x)]} \exp(-u) du \quad (20)$$

Then, by using arguments in Giorgio et al. (2015a), the conditional pdf of X_t can be expressed in an analytical form, which does not involve numerical integration:

$$f_{X_t}(x|w_t) = \frac{d\eta(t, t+x)}{dx} \frac{1}{\Gamma[\eta(t, t+x)]} \times \left\{ IG[g(w_t, D); \eta(t, t+x)] \left(\psi[\eta(t, t+x)] \right. \right.$$

$$\left. - \ln[g(w_t, D)] \right) + \sum_{k=0}^{\infty} \frac{(-1)^k [g(w_t, D)]^{\eta(t, t+x)+k}}{[\eta(t, t+x) + k]^2 k!} \left. \right\}, \quad (21)$$

where $\psi(z)$ denotes the digamma function. Likewise, from (21), the pdf of the lifetime X of a new unit is given by:

$$f_X(x) = \frac{d\eta(x)}{dx} \frac{1}{\Gamma[\eta(x)]} \left\{ IG[g(D); \eta(x)] (\psi[\eta(x)] - \ln[g(D)]) + \sum_{k=0}^{\infty} \frac{(-1)^k [g(D)]^{\eta(x)+k}}{[\eta(x) + k]^2 k!} \right\}. \quad (22)$$

Finally, the mean $E\{X_t|w_t\}$ of the RUL and the mean lifetime $E\{X\}$ of a new item are given by, respectively:

$$E\{X_t|w_t\} = \int_0^{\infty} R_t(x|w_t) dx$$

$$E\{X\} = \int_0^{\infty} R(x) dx .$$

4. The Estimation Procedure

Let us suppose that the degradation level of m identical units, which operate under the same conditions, is measured by performing periodic inspections, where both the number of measurements n_i ($i = 1, \dots, m$) and the ages of the units at the inspection epochs could vary from unit to unit. Let $w_{i,j}$ and $t_{i,j}$ ($i = 1, \dots, m; j = 1, \dots, n_i$) denote the degradation level and the age of the unit i at the epoch of the j -th inspection, respectively. From (1), the conditional pdf of the wear increment $\Delta W(t_{i,j-1}, t_{i,j})$ accumulated by the unit i during the inspection interval $(t_{i,j-1}, t_{i,j})$, given the state $W(t_{i,j-1}) = w_{i,j-1}$ at the beginning of the interval, is:

$$f_{\Delta W(t_{i,j-1}, t_{i,j})}(\delta_{i,j} | w_{i,j-1}) = g'(w_{i,j-1} + \delta_{i,j}) \frac{(\Delta g_{i,j})^{\Delta \eta_{i,j}-1}}{\Gamma[\Delta \eta_{i,j}]} \exp[-\Delta g_{i,j}],$$

$$0 < \delta_{i,j} < w_{lim} - w_{i,j-1}, \quad (23)$$

where $\delta_{i,j} = w_{i,j} - w_{i,j-1}$, $\Delta g_{i,j} = g(w_{i,j-1}, w_{i,j-1} + \delta_{i,j})$, $\Delta \eta_{i,j} = \eta(t_{i,j-1}, t_{i,j})$, with $w_{i,0} = t_{i,0} = 0$ for all i . Then, the log-likelihood function relative to the observed data $\mathbf{w} = (w_{1,1}, \dots, w_{1,n_1}, \dots, w_{m,1}, \dots, w_{m,n_m})$ is:

$$\ell(\mathbf{w}; \boldsymbol{\theta}, w_{lim}) = \sum_{i=1}^m \sum_{j=1}^{n_i} \ln (f_{\Delta W(t_{i,j-1}, t_{i,j})}(\delta_{i,j} | w_{i,j-1})), \quad (24)$$

where $\boldsymbol{\theta}$ denotes the vector of parameters which, together to w_{lim} , index the age and state functions. For example, if the age function is the power-law function $\eta(t) = (t/a)^b$ and the state function is one of those in (3)-(5), then $\boldsymbol{\theta} = (a, b, \beta)$.

The maximum likelihood (ML) estimates $\hat{\boldsymbol{\theta}}$ and \hat{w}_{lim} of the process parameters can be easily obtained by numerically maximizing the log-likelihood function (24) with respect to $\boldsymbol{\theta}$ and w_{lim} .

In several circumstances, however, previous experiences and/or physical considerations on the degradation phenomenon allow the analyst to fix a lower limit or an interval of plausible values for w_{lim} . In this case, a constrained maximization procedure of the log-likelihood (24) has to be used in order to satisfy the above constraints.

Once the vector of unknown parameters has been estimated, the ML estimate of any quantity of interest, say $\phi(\theta, w_{lim})$, can be easily obtained by substituting $\hat{\theta}$ and \hat{w}_{lim} to θ and w_{lim} in $\phi(\theta, w_{lim})$. In particular, by using this approach, one can easily obtain the ML estimates of the conditional distribution of the RUL, the mean of the RUL, the residual reliability, and the conditional distribution of the degradation growth over a future time interval.

5. The Numerical Application

Let now consider the wear measurements, given in Table 1, of the liners of the 8-cylinder SULZER engine which equips a cargo ship of the Grimaldi Lines, which operates on the same routes under homogeneous operating conditions. The dataset consists of a total of 23 wear measurements gathered via ad hoc inspections carried out between 1994 and 2004. At each inspection, the wear of the inspected liner is measured by positioning a micrometer inside a predetermined hole positioned at the top dead center of the liner. Each datum consists of the age of a liner at the measurement epoch and of the wear measurement. Because the micrometer accuracy is of 0.05 mm, each measure is rounded up to the nearest multiple of 0.05 mm. A liner is conventionally considered failed when its wear level passes the threshold limit $D = 4$ mm.

Table 1. Wear $w_{i,j} = W(t_{i,j})$ [mm] accumulated by liner i up to the inspection time $t_{i,j}$ [hours].

i	$w_{i,1}$	$t_{i,1}$	$w_{i,2}$	$t_{i,2}$	$w_{i,3}$	$t_{i,3}$	$w_{i,4}$	$t_{i,4}$
1	0.90	11,300	1.30	14,680	2.85	31,270		
2	1.50	11,300	2.00	21,970				
3	1.00	12,300	1.35	16,300				
4	1.90	14,810	2.25	18,700	2.75	28,000		
5	1.20	10,000	2.75	30,450	3.05	37,310		
6	0.50	6,860	1.45	17,200	2.15	24,710		
7	0.40	2,040	2.00	12,580	2.35	16,620		
8	0.50	7,540	1.10	8,840	1.15	9,770	2.10	16,300

This wear process was previously analyzed in Giorgio et al. (2015b) under a transformed gamma (TG) process with unbounded state function, say the power-law function $g(w) = (w/\alpha)^\beta$, and two different age functions, say the power-law function $\eta(t) = (t/a)^b$ and the exponential function $\eta(t) = ab[\exp(t/b) - 1]$.

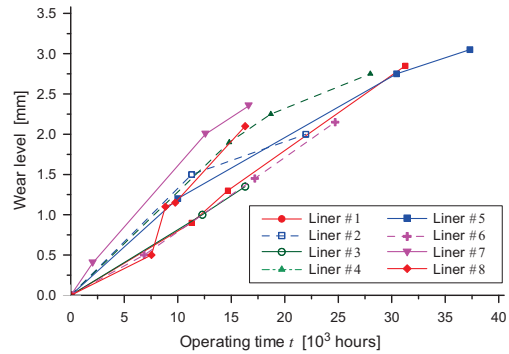


Fig. 3. Observed paths of the liner wear (the measured points are linearly connected for graphical convenience).

The best fit for these data was provided by the TG process with power-law age function, but although this TG process is able to fit quite well the empirical mean of the wear process (see Figure 7 of Giorgio et al. (2015b)), it is not able to fit adequately the empirical variance (see Figure 8 of Giorgio et al. (2015b)), in particular when the empirical variance decreases quickly.

On the other side, the considered wearing phenomenon possesses the following two interesting features: *a*) physical considerations suggest that the wear can not growth up to the thickness of the liners (i.e., 100 mm), and *b*) the same physical considerations alone do not allow to determine the exact value of the upper bound. Thus, also considered—that the proposed BTG process can describe the behavior of the empirical variance given in Figure 8 of Giorgio et al. (2015b), the liner wear data are here analyzed under the proposed BTG process. In particular, the power-law age function $\eta(t) = (t/a)^b$ and the three bounded state functions (3)-(5) are considered.

In addition, given that previous experiences suggest that the liner wear can approach the value of 4.3 mm, the process parameters are estimated under the constraint $w_{lim} \geq 4.3$ mm.

It is worth to note that, in this experimental situation, assuming that the upper bound of the BTG process coincides with the thickness of the liner is practically equivalent to use unbounded TG process.

In Table 2 the ML estimates of the parameters of the three considered BTG processes, the corresponding estimated log-likelihood $\hat{\ell}$ and Akaike information criterion (AIC) value (see, Akaike (1974)) are provided, and compared to the estimates obtained in Giorgio et al. (2015b) under the TG process. Note that the BTG process with bounded state function $g_l(w)$, $l = 1, 2, 3$, is here denoted as “BTG l ”.

Table 2. Estimation results under the BTG and the TG processes.

Process	\hat{a} [h]	\hat{b}	\hat{w}_{lim} or $\hat{\alpha}$ [mm]	$\hat{\beta}$	$\hat{\tau}$	AIC
BTG1	1295	1.106	4.3	31.36	2.657	2.686
BTG2	2682	1.434	4.363	18.62	3.843	0.314
BTG3	1504	1.182	4.3	21.67	3.837	0.327
TG	5107	1.701	2.321	0.750	0.590	6.820

The estimation results given in Table 2 show that all the proposed BTG processes fit the liner wear data better than the TG process, because the corresponding AIC values are smaller than the AIC value relative to the TG process. In addition, among the three BTG processes, those that provide the best fit are the ones with state functions $g_2(w)$ given in (4) and $g_3(w)$ given in (5). The difference between the corresponding AIC values is so small that it does not allow a final choice between the models BTG2 and BTG3.

In Figures 4 and 5 the ML estimates of the mean $E\{W(t)\}$ and variance $V\{W(t)\}$ of the wear process under the TG, the BTG2, and the BTG3 processes are compared to the empirical estimates.

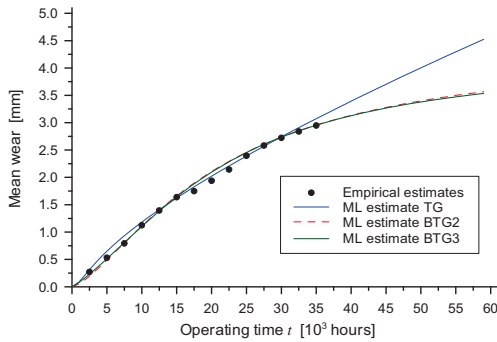


Fig. 4. Empirical and ML estimates of the mean wear $E\{W(t)\}$ under the TG, the BTG2, and the BTG3 processes.

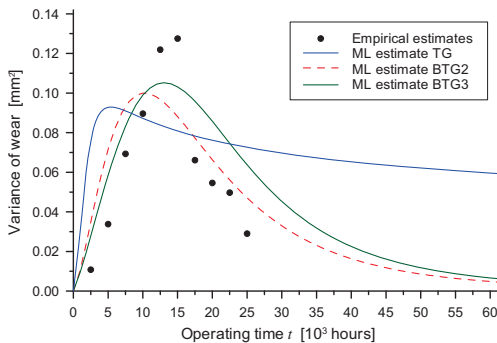


Fig. 5. Empirical and ML estimates of the variance $V\{W(t)\}$ under the TG, BTG2, and the BTG3 processes.

Note that, since the inspection times generally differ, and hence the wear measures generally refer to different operating times of the liners, the empirical estimates of mean and variance were obtained by using the interpolation procedure at selected equispaced times used in Giorgio et al. (2015b).

From Figure 4 we have that the two BTG processes provide almost the identical estimate of the mean wear $E\{W(t)\}$, and fit the empirical mean a little better than the TG process, in particular at large time t where the TG mean is not able to bend down. For its part, Figure 5 shows that the two BTG processes, unlike the TG one, are able to fit adequately both the initial growth and the subsequent rapid reduction of the empirical variance.

In Figure 6 the ML estimates of the reliability of a new liner under the TG, the BTG2, and the BTG3 processes are plotted. We note that the TG process has greatly underestimated the unit reliability, and consequently also the mean lifetime. Indeed, the ML estimates of the mean lifetime $E\{X\}$ is equal to 109,994 h and 131,629 h under the BTG2 and the BTG3 processes, respectively, whereas under the TG process it is only equal to 49,954 h.

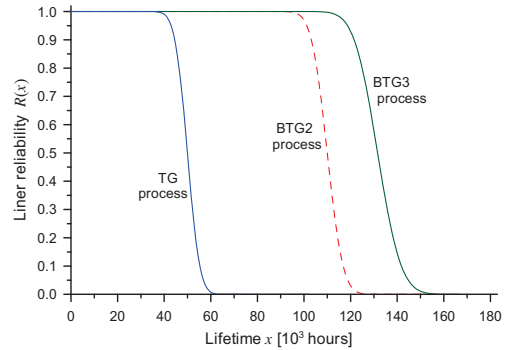


Fig. 6. ML estimates of the reliability $R(x)$ of a new liner under the TG, BTG2, and BTG3 processes.

In Figure 7 the ML estimates of the conditional pdf $f_{X_t}(x|w_t)$ of the RUL X_t of liner #5, given the current wear level $w_t = 3.05$ mm at the current age $t = 37,310$ h, under the TG, BTG2, and BTG3 processes are depicted. From these plots we note that the TG process has greatly underestimated the RUL of the liner, whose estimated mean value $E\{X_t|w_t\}$ is equal to 65,800 h and 94,537 h under the BTG2 and the BTG3 processes, respectively, whereas under the TG process it is only equal to 14,965 h. It should be noted that similar results have been obtained for the other liners.

Finally, Figure 8 provides the ML estimates of the conditional pdf $f_{\Delta W(t,t+\Delta t)}(\delta|w_t)$ of the wear increment $\Delta W(t, t + \Delta t)$ of liner #1 during the future time interval of width $\Delta t = 10,000$ h, given the current wear level $w_t = 2.85$ mm at the current age $t = 31,270$ h, under the TG, BTG2, and BTG3 processes.

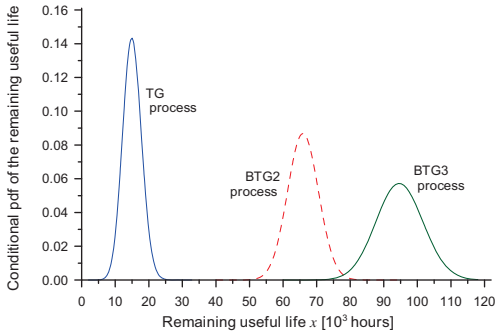


Fig. 7. ML estimates of the conditional pdf $f_{X_t}(x|w_t)$ of the remaining useful life X_t of liner #5, given $w_t = 3.05$ mm at $t = 37,310$ h, under the TG, BTG2, and BTG3 processes.

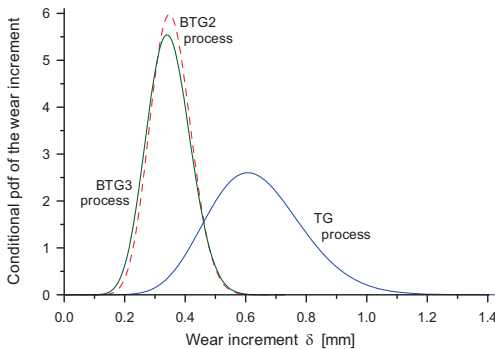


Fig. 8. ML estimates of the conditional pdf $f_{\Delta W(t,t+\Delta t)}(\delta|w_t)$ of the wear increment of liner #1 during the interval (31,270 h, 31,270+10,000 h), given $w_t = 2.85$ mm, under the TG, BTG2, and BTG3 processes.

From these plots we note that the TG process has greatly overestimated the future wear growth of the liner, and that, differently from the reliability estimates in Figures 6 and 7, the two BTG processes provide almost the same conditional distribution. Indeed, the estimated mean wear increment $E\{\Delta W(t, t + \Delta t)|w_t\}$ is equal to 0.353 mm and 0.346 mm under the BTG2 and the BTG3 processes, respectively, whereas it is equal to 0.637 mm under the TG process. Also in this case, similar results have been obtained for the other liners.

6. Conclusions

This paper investigated the potentiality of the transformed gamma process in modelling degradation processes where the degradation can not grow indeterminately, due to geometric characteristics of the degrading units or the nature itself of the degradation mechanism. At this aim, three different bounded state functions are proposed and used to define the bounded, state-dependent, transformed gamma process. The main characteristics of this process are then illustrated. Finally, a set of real data, consisting of

wear measurements of cylinder liners equipping a diesel engine for marine propulsion, is analyzed under the proposed process and the corresponding estimation results are compared to those provided by an unbounded transformed gamma process previously adopted to analyze the same wear data. This comparative study involving the unit reliability and the wear increment during future time interval shows that the adoption of a degradation process that does not take into account the existence of an upper bound for the degradation growth can lead to very poor reliability estimates and life predictions.

Acknowledgement

The authors would like to thank the anonymous reviewers for their valuable comments and suggestions that allowed to improve the quality of the paper.

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