Cost-Effective Planning Reliability-Based Inspections of Fatigued Structures in the Case of Log-Location-Scale Distributions of Lifetime under Parametric Uncertainty

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Many important fatigued structures (for instance, Transportation Systems and Vehicles: aircraft, space vehicles, trains, ships; Civil Structures: bridges, dams, tunnels; and so on) for which extremely high reliability is required are maintained by in-service inspections to prevent the reliability degradation due to fatigue damage. However, temporal transition of the reliability is significantly affected by the inspection strategy selected. Thus, to keep structures reliable against fatigue damage by inspections, it is clearly important in engineering to examine the optimal inspection strategy. If reliability is the only criterion based on which the effect of inspection is quantitatively assessed we inevitably arrive at the conclusion that it is best to make as many inspections as possible. However, this is unfeasible in a real-life engineering environment, since it causes not only the deterioration of the system availability, but also an increase in costs. Therefore, it is more effective to determine inspection strategy based upon a cost-based criterion, which enables us to make a quantitative assessment from both reliability-based and cost-based points of view. In this paper, sequential inspections are obtained to ensure that the conditional fatigue reliability is at the required level. Frequent inspection leads to a high cost and infrequent inspection will lead to low fatigue reliability of the system upon demand. Although the cost might be an issue in this type of analysis, the focus here is to meet the fatigue reliability requirement with an appropriate time to next inspection. The sequential inspection procedure and decision making procedure studied in this paper allows an appropriate level of fatigue reliability to be reached with minimum cost as well. Furthermore, we do not assume the distribution of system lifetime to be completely known which is usually the case. The invariant embedding and averaging approach (IEAA) proposed in this paper represents a simple and computationally attractive statistical method based on the constructive use of the invariance principle in mathematical statistics. It allows one to improve the decision-making process under parametric uncertainty by removing unknown parameters from the problem and using the past lifetime data as completely as possible. IEAA includes the following 3 steps: Step 1. Invariant embedding of a sample statistic in the decision criterion to construct a *pivotal quantity* (or simply a *pivot*) to isolate the unknown parameter (*the pivot's probability distribution does not depend on the unknown parameters*); Step 2. The decision criterion is averaged over the pivotal quantities to exclude the unknown parameters from the problem; Step 3. Decision-making process (it is used when the unknown parameters are excluded from the decision criterion).

Keywords: Fatigued structure, fatigue damage, log-location-scale distribution, parametric uncertainty, reliability-based inspections, cost-effective planning.

1. Introduction

Fatigue is one of the most important problems of aircraft arising from their nature as multiple-component structures, subjected to random dynamic loads. Fatigue damage increases with the number of applied loading cycles in a cumulative manner and can lead to fracture and failure of the considered part. An example of cracked stringer clip from B737 aircraft is given on Fig. 1 (Nechval 2008). A post-failure photograph of one of the F-16 479 bulkhead test

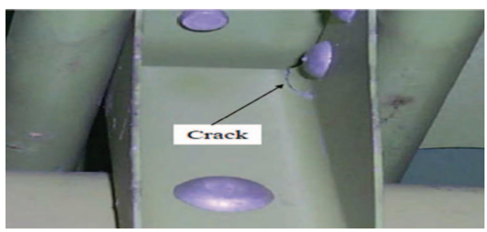

Fig. 1. Example of cracked stringer clip from B737 aircraft. Source: Nechval (2008).

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components (Fig. 2) indicates the location of fatigue crack initiation at the radius between the bulkhead and one of the two vertical tail attach pads (Nechval 2008).

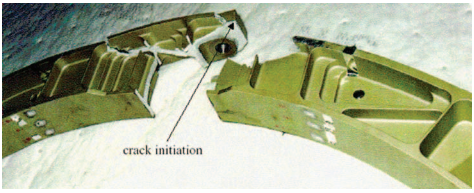

Fig. 2. F-16 479 bulkhead test specimen number -7B. Source: Nechval (2008).

In Fig. 3, it is given micrograph showing one of the cracks detected in the bladed disk assembly of the High Pressure Turbine for over 17,000 cycles (Nechval 2008).

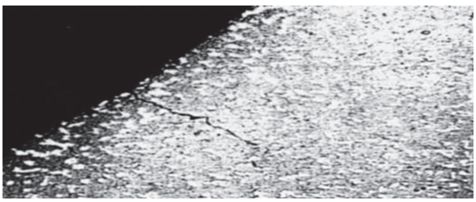

Fig. 3. Micrograph showing one of the detected cracks. Source: Nechval (2008).

From an engineering standpoint the fatigue life of a structure (or component) consists of two periods (this concept is shown schematically in Fig. 4).

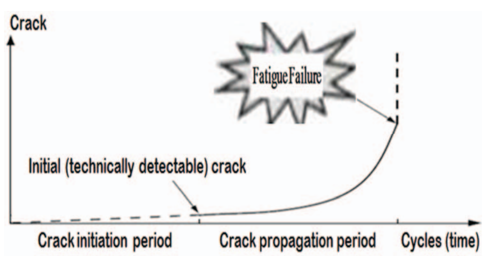

Fig. 4. Schematic fatigue crack growth curve (Crack initiation and propagation periods).

The intervals between inspections of fatigued structures should be gradually smaller in order to restrain the reliability degradation by repeated inspections. Therefore, we need to construct the inspection strategy by paying attention to this case. Barlow et al. (1963) tackled this problem by assuming a known, fixed cost of making an inspection and a known fixed cost per unit time due to undetected failure. They then found a sequence of inspection times for which the expected cost is a minimum. Their results have been extended by various authors (Luss and Kander 1974; Sengupta 1977). Unfortunately, it is difficult to compute optimal checking procedures numerically, because the computations are repeated until the procedures are determined to the required degree by changing the first check time. To avoid this, Munford and Shahani (1972) suggested a sub-optimal (or nearly optimal) but computationally easier inspection policy. This policy was used for Weibull and gamma failure distribution cases (Munford and Shahani 1973; Tadikamalla 1979). Numerical comparisons among certain inspection policies are given by Munford (1981) for the case of Weibull failure times. The different cases of inspection planning for fatigued structures were considered by Nechval et al. (2003, 2004, 2008, 2009, 2011, 2015, 2017), Sengupta (1977), Straub and Faber (2005).

In this paper, the new technique of cost-effective planning reliability-based inspections of fatigued structures in the case of log-location-scale distributions of lifetime is proposed to construct more accurate reliability-based inspections of fatigued structures (with decreasing intervals as alternative to constant intervals often used in practice for convenience in operation) for the fatigue crack initiation period under parametric uncertainty. It is conceptually simple and easy to use.

2. Two-Parameter Weibull Distribution of Lifetime to Detection of Fatigue Crack

The two-parameter Weibull distribution is one of the most widely used life distributions in reliability analysis. This distribution is very flexible, and can, through an appropriate choice of parameters, model many types of failure rate behaviors. It has wide applications in diverse disciplines.

In this paper, we suppose that the initial (technically detectable) fatigue crack (damage) can only be detected through inspection.

Let $X_1 \le \ldots \le X_r$ be the first r ordered observations of a random variable X (lifetime to detection of fatigue crack) from a sample of size n from a two-parameter Weibull distribution with the probability density function (pdf),

$$f_\theta(x) = \frac{\delta}{\beta}\left(\frac{x}{\beta}\right)^{\delta-1} \exp\left[-\left(\frac{x}{\beta}\right)^{\delta}\right],$$

$$x > 0,\ \ \beta > 0,\ \ \delta > 0 \#(1)$$

and cumulative distribution function (cdf),

$$F_\theta(x) = 1 - \exp\left[-\left(\frac{x}{\beta}\right)^{\delta}\right],$$

$$x > 0,\ \ \beta > 0,\ \ \delta > 0 \#(2)$$

indexed by scale and shape parameters β and δ, where $\theta = (\beta, \delta)$. It is assumed that the parameters β and δ are unknown. The MLE's of the Weibull parameters β and δ are given by

$$\widehat{\beta} = \left(r^{-1} \left(\sum_{i=1}^{r} x_i^{\widehat{\delta}} + (n-r) x_r^{\widehat{\delta}} \right) \right)^{1/\widehat{\delta}} \#(3)$$

and

$$\widehat{\delta} = \left[\begin{array}{l} \left(\sum_{i=1}^{r} x_i^{\widehat{\delta}} \ln x_i + (n-r) x_r^{\widehat{\delta}} \ln x_r \right) \\ \times \left(\sum_{i=1}^{r} x_i^{\widehat{\delta}} + (n-r) x_r^{\widehat{\delta}} \right)^{-1} - \frac{1}{r} \sum_{i=1}^{r} \ln x_i \end{array} \right]^{-1} . \#(4)$$

In terms of the Weibull variates, we have that

$$V_1 = \left(\frac{\widehat{\beta}}{\beta} \right)^{\delta}, \quad V_2 = \frac{\delta}{\widehat{\delta}}, \quad V_3 = \left(\frac{\widehat{\beta}}{\beta} \right)^{\widehat{\delta}} \#(5)$$

are pivotal quantities. The probability density functions of the pivotal quantities do not depend on the parameters.

It can be shown that the joint pdf of the pivotal quantities

$$V_1 = \left(\frac{\widehat{\beta}}{\beta} \right)^{\delta}, \quad V_2 = \frac{\delta}{\widehat{\delta}}, \#(6)$$

conditional on fixed

$$\mathbf{Z}^{(r)} = (Z_1, \ldots, Z_r), \#(7)$$

where

$$Z_i = \left(\frac{X_i}{\widehat{\beta}} \right)^{\widehat{\delta}}, \quad i = 1, \ldots, r, \#(8)$$

are ancillary statistics, any r–2 of which form a functionally independent set, $\widehat{\beta}$ and $\widehat{\delta}$ are the maximum likelihood estimates for β and δ, respectively, based on the first r ordered observations $X_1 \le \ldots \le X_r$ from a sample of size n from the two-parameter Weibull distribution "Eq. (1)", which can be found from solution of "Eq. (3)" and "Eq. (4)", is given by

$$f_n(v_1, v_2 \mid \mathbf{z}^{(r)}) = \frac{v_1^{r-1}}{\Gamma(r)} \left[\sum_{i=1}^{r} z_i^{v_2} + (n-r) z_r^{v_2} \right]^r$$
$$\times \exp\left(-v_1 \left[\sum_{i=1}^{r} z_i^{v_2} + (n-r) z_r^{v_2} \right] \right)$$
$$\times \frac{1}{\vartheta(\mathbf{z}^{(r)})} v_2^{r-2} \prod_{i=1}^{r} z_i^{v_2} \left[\sum_{i=1}^{r} z_i^{v_2} + (n-r) z_r^{v_2} \right]^{-r}$$
$$= f_n(v_1 \mid \mathbf{z}^{(r)}, v_2) f_n(v_2 \mid \mathbf{z}^{(r)}),$$
$$v_1 \in (0, \infty), \quad v_2 \in (0, \infty), \#(9)$$

where

$$f_n(v_1 \mid \mathbf{z}^{(r)}, v_2) = \frac{1}{\Gamma(r)} \left[\sum_{i=1}^{r} z_i^{v_2} + (n-r) z_r^{v_2} \right]^r$$
$$\times v_1^{r-1} \exp\left(-v_1 \left[\sum_{i=1}^{r} z_i^{v_2} + (n-r) z_r^{v_2} \right] \right),$$
$$v_1 \in (0, \infty), \#(10)$$

$$f_n(v_2 \mid \mathbf{z}^{(r)}) = \frac{1}{\vartheta(\mathbf{z}^{(r)})}$$
$$\times v_2^{r-2} \prod_{i=1}^{r} z_i^{v_2} \left(\sum_{i=1}^{r} z_i^{v_2} + (n-r) z_r^{v_2} \right)^{-r},$$
$$v_2 \in (0, \infty), \#(11)$$

$$\vartheta(\mathbf{z}^{(r)})$$
$$= \int_0^{\infty} v_2^{r-2} \prod_{i=1}^{r} z_i^{v_2} \left(\sum_{i=1}^{r} z_i^{v_2} + (n-r) z_r^{v_2} \right)^{-r} dv_2 \#(12)$$

is the normalizing constant.

3. Reliability-Based Inspection Strategy under Parametric Certainty

The inspection strategy is based on the conditional reliability of the structure. It is given as follows. Fix $0 < \gamma < 1$ and let

$$\tau_1 = \arg\ (\Pr\{X > \tau_1\} = \gamma), \#(13)$$
$$\tau_j = \arg\ (\Pr\{X > \tau_j \mid X > \tau_{j-1}\} = \gamma),$$
$$\tau_j > \tau_{j-1}, \quad j \ge 2, \#(14)$$

where $\{\tau_j\}_{j=1,\ 2,\ \ldots}$ are inspection times, X is a random variable representing the lifetime of the component (structure). This is named as 'reliability-based inspection strategy'. The above inspection strategy makes use of the information about the remaining life that is inherent in the sequence of previous inspection times. The value of γ can be seen as 'minimum fatigue reliability required' (or 'fatigue reliability index') during the next period when the structure was still operational at last inspection time, that is, in other words, the conditional probability that the failure (fatigue crack) occurs in the time interval (τ_{j-1}, τ_j) without failure at time τ_{j-1} is always assumed $1-\gamma$. It is clear that if F_θ, the structure lifetime distribution with the parameter θ (in general, vector), is continuous and strictly increasing, the definition of the inspection strategy is equivalent to

$$\tau_j = \arg(\overline{F}_\theta(\tau_j) = \gamma^j), \quad j = 1,\ 2,\ 3,\ \ldots, \#(15)$$

where

$$\overline{F}_\theta(\tau_j) = 1 - F_\theta(\tau_j). \#(16)$$

4. Numerical Example 1

If the lifetime X to detection of fatigue crack follows the two-parameter Weibull distribution "Eq. (1)", then it follows from "Eq. (15)" that

$$\tau_j = \beta\left[j\ln\left(\frac{1}{\gamma}\right)\right]^{1/\delta}, \quad j=1,\ 2,\ 3,\ ....\#(17)$$

For illustration, if $\beta = 2000$, $\delta = 4$, $\gamma = 0.95$, $\tau_0=0$, the intervals between inspections ($\Delta_j=\tau_j-\tau_{j-1}$) are shown in Fig. 5.

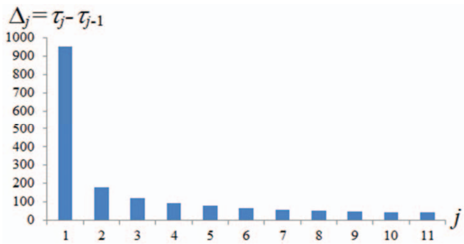

Fig. 5. Illustration of the reliability-based inspection strategy.

5. Total Expected Cost per Inspection Cycle under Parametric Certainty

If it is known that each inspection costs c_1 and the cost of leaving an undetected failure (fatigue crack) is c_2 per unit time, then the total expected cost per inspection cycle is given by

$$C_\theta(\tau(\gamma)) = \sum_{j=1}^{\infty}\int_{\tau_{j-1}}^{\tau_j}[jc_1 + c_2(\tau_j - x)]f_\theta(x)dx$$

$$= c_1\sum_{j=1}^{\infty} j\,[F_\theta(\tau_j)-F_\theta(\tau_{j-1})]$$

$$+\, c_2\sum_{j=1}^{\infty}\tau_j[F_\theta(\tau_j)-F_\theta(\tau_{j-1})] - c_2\int_0^\infty x f_\theta(x)dx$$

$$= c_1\sum_{j=0}^{\infty}\bar{F}_\theta(\tau_j) + c_2\sum_{j=1}^{\infty}\tau_j[\bar{F}_\theta(\tau_{j-1})-\bar{F}_\theta(\tau_j)]$$

$$-c_2E_\theta\{X\}$$

$$= c_1\sum_{j=0}^{\infty}\gamma^j + c_2\sum_{j=1}^{\infty}\tau_j[\gamma^{j-1}-\gamma^j] - c_2E_\theta\{X\}$$

$$= c_2\left(\frac{\vartheta}{1-\gamma} + \frac{1-\gamma}{\gamma}\sum_{j=1}^{\infty}\bar{F}_\theta^{-1}(\gamma^j)\gamma^j - E_\theta\{X\}\right), \#(18)$$

where $f_\theta(x)$ "Eq. (1)" is the probability density function of the structure lifetime X to detection of fatigue crack,

$$\tau_0 = 0, \quad \tau_j = \bar{F}_\theta^{-1}(\gamma^j), \quad j=1,\ 2,\ 3,\ ...,\#(19)$$

$$E_\theta\{X\} = \int_0^\infty x f_\theta(x)dx$$

$$= \int_0^\infty \bar{F}_\theta(x)dx = \beta\,\Gamma\left(1+\frac{1}{\delta}\right), \#(20)$$

$$\vartheta = c_1/c_2.\#(21)$$

6. Minimization of the Total Expected Cost per Inspection Cycle via the Reliability Index γ

The problem is to choose the reliability index γ such that the total expected cost per inspection cycle, $C(\tau(\theta,\gamma))$, as defined in "Eq. (18)", is minimized. The optimal value of the reliability index γ is determined as

$$\gamma^* = \arg\min_{0<\gamma<1} C(\tau(\theta,\gamma))$$

$$= \arg\min_{0<\gamma<1} c_2\left(\begin{array}{c}\dfrac{\vartheta}{1-\gamma} + \dfrac{1-\gamma}{\gamma} \\ \times \displaystyle\sum_{j=1}^{\infty}\bar{F}_\theta^{-1}(\gamma^j)\gamma^j - E_\theta\{X\}\end{array}\right). \#(22)$$

The optimal value of γ has to be determined numerically, and an accurate minimization of $C(\tau(\theta,\gamma))$ is made easy by the rapid convergence of the infinite sum in "Eq. (18)". Note that the computation of optimal γ needs θ and c_1/c_2. The relevance of the cost ratio c_1/c_2 rather than the individual costs c_1, c_2 may be a help in some cases; it may be that the relative cost c_1/c_2 is available while the individual costs c_1, c_2 are not available. Given γ, the inspection times τ_1, τ_2, … can be easily computed from equation "Eq. (19)".

7. Index of Improvement Percentage in Effectiveness of Inspection Strategy

The index of improvement percentage in effectiveness of the optimal inspection strategy (with $\gamma = \gamma^*$) as compared with the standard inspection strategy (with $\gamma = \gamma_{\text{st}}$) is given by

$$I_{\text{imp.per}}(\gamma^*,\gamma_{\text{st}})$$

$$= \frac{C_\theta(\tau(\gamma_{\text{st}})) - C_\theta(\tau(\gamma^*))}{C_\theta(\tau(\gamma_{\text{st}}))}100\%. \#(23)$$

8. Numerical Example 2

Under the conditions of Numerical Example 1, suppose for simplicity, but without loss of generality, that δ=1. Then it follows from "Eq. (18)" and "Eq. (19)" that

$$C_\theta(\tau(\gamma)) = c_2\beta\left(\frac{1}{1-\gamma}\left[\frac{\vartheta}{\beta} + \ln\left(\frac{1}{\gamma}\right)\right] - 1\right), \#(24)$$

and

$$\tau_j = \bar{F}_\theta^{-1}(\gamma^j) = \beta\left[j\ln\left(\frac{1}{\gamma}\right)\right], \quad j=1,\ 2,\ 3,\ ...,\#(25)$$

respectively.

Let us assume that $\vartheta=c_1/c_2=1/10$, then it follows from "Eq. (24)" that

$$\gamma^* = \arg\min_{0<\gamma<1} C_\theta(\tau(\gamma))$$

$$= \arg\min_{0<\gamma<1} c_2\beta\left(\frac{1}{1-\gamma}\left[\frac{\vartheta}{\beta}+\ln\left(\frac{1}{\gamma}\right)\right]-1\right) = 0.99 \#(26)$$

and

$$C_\theta(\tau(\gamma^*)) = 200.6717. \#(27)$$

If the standard fatigue reliability index $\gamma = \gamma_{st} = 0.95$ is used, then

$$C_\theta(\tau(\gamma_{st})) = c_2\beta\left(\frac{1}{1-\gamma_{st}}\left[\frac{\vartheta}{\beta}+\ln\left(\frac{1}{\gamma_{st}}\right)\right]-1\right)$$

$$= 537.3178. \#(28)$$

It follows from (52) that the index of improvement percentage in effectiveness of the optimal inspection strategy (with $\gamma = \gamma^* = 0.99$) as compared with the standard inspection strategy (with $\gamma = \gamma_{st} = 0.95$) is given by

$$I_{\text{imp.per}}(\gamma^*, \gamma_{st}) = \frac{C_\theta(\tau(\gamma_{st})) - C_\theta(\tau(\gamma^*))}{C_\theta(\tau(\gamma_{st}))} 100\%$$

$$= 62.65307\%. \#(29)$$

Fig. 6 depicts the relationship between $C(\tau(\theta,\gamma))$ and γ.

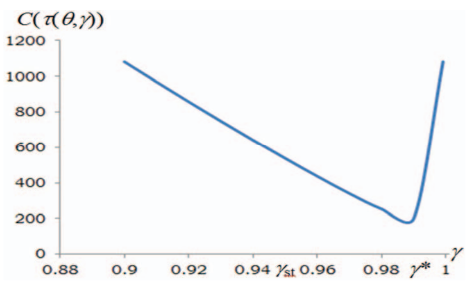

Fig. 6. Variation of $C_\theta(\tau(\gamma))$ as a function of γ.

9. Unbiased Reliability-Based Inspection Strategy when the Scale Parameter β is Unknown

When the scale parameter β is unknown, then the unbiased reliability-based inspection strategy is determined as follows.

Theorem 1. Let $X_1 \le \ldots \le X_r$ be the first r ordered observations from a past random sample of size n of lifetimes to detection of fatigue crack from the fatigued structures (components) of the same type, which follow the two-parameter Weibull distribution "Eq. (1)" where the scale parameter β is unknown. Then the unbiased reliability-based inspection strategy for a new fatigued structure (component) of the same type is given by

$$\tau_j^{\text{unb}} = [r(\gamma^{-j/r}-1)]^{1/\delta}\widehat{\beta}, \quad j=1,\ 2,\ 3,\ \ldots, \#(30)$$

where

$$\widehat{\beta} = \left(r^{-1}\left(\sum_{i=1}^{r} x_i^\delta + (n-r)x_r^\delta\right)\right)^{1/\delta}. \#(31)$$

Proof. Using the pivotal quantity averaging approach (PQAA) based on invariant embedding of sample statistics in the performance index (Nechval *et al.* 2008), the unbiased reliability-based inspection strategy "Eq. (30)" can be obtained as follows.

$$\overline{F}_\theta(\tau_j) = \exp\left[-\left(\frac{\tau_j}{\beta}\right)^\delta\right]$$

$$= \exp\left[-\left(\frac{\tau_j}{\widehat{\beta}}\right)^\delta\left(\frac{\widehat{\beta}}{\beta}\right)^\delta\right] = \exp\left[-\rho_j V\right], \#(32)$$

where an ancillary factor

$$\rho_j = \left(\frac{\tau_j}{\widehat{\beta}}\right)^\delta, \#(33)$$

the pivotal quantity

$$V = \left(\frac{\widehat{\beta}}{\beta}\right)^\delta \sim f_r(v) = \frac{r^r}{\Gamma(r)} v^{r-1}\exp(-rv)$$

$$v \in (0,\infty). \#(34)$$

It follows from "Eq. (32)" that

$$E_\theta\left\{\overline{F}_\theta(\tau_j)\right\} = E\left\{\exp\left[-\rho_j V\right]\right\} = \int_0^\infty \exp\left[-\rho_j v\right] f_r(v)dv$$

$$= \int_0^\infty \frac{r^r}{\Gamma(r)} v^{r-1}\exp(-v[r+\rho_j])dv = \frac{r^r}{[r+\rho_j]^r}. \#(35)$$

Thus, the unbiased reliability-based inspection strategy is given by

$$\tau_j^{\text{unb}} = \arg\left(E_\theta\left\{\overline{F}_\theta(\tau_j)\right\} = \gamma^j\right)$$

$$= [r(\gamma^{-j/r}-1)]^{1/\delta}\widehat{\beta}, \quad j=1,\ 2,\ 3,\ \ldots. \#(36)$$

It is clear that

$$E_\theta\left\{\overline{F}_\theta(\tau_j^{\text{unb}})\right\} = E_\theta\left\{\overline{F}_\theta([r(\gamma^{-j/r}-1)]^{1/\delta}\widehat{\beta})\right\}$$

$$= E\left\{\exp\left(-[r(\gamma^{-j/r}-1)]V\right)\right\} = \gamma^j. \#(37)$$

This ends the proof.

10. Total Estimated Cost per Inspection Cycle when the Scale Parameter β is Unknown

Theorem 2. If (under conditions of Theorem 1) the unbiased reliability-based inspection strategy (30) is used, where each inspection costs c_1 and the cost of leaving an undetected failure (fatigue crack) is c_2 per unit time, then the total estimated cost per inspection cycle is given by

$$E\left\{C_{\widehat{\beta},\delta}(\tau^{\mathrm{unb}}(\gamma))\right\}$$

$$=c_2\begin{pmatrix}\vartheta/(1-\gamma)+r^{1/\delta}\widehat{\beta}(1/\gamma-1)\\ \times\sum_{j=1}^{\infty}\gamma^j[\gamma^{-j/r}-1]^{1/\delta}-\frac{r\widehat{\beta}}{r-1}\Gamma\left(1+\frac{1}{\delta}\right)\end{pmatrix}.\#(38)$$

Proof. It follows from "Eq. (18)" that $C_\theta(\tau(\gamma))$ is given by

$$C_\theta(\gamma,\tau(\gamma))=c_2\left(\vartheta\sum_{j=0}^{\infty}\bar{F}_\theta(\tau_j)+\sum_{j=1}^{\infty}\tau_j[\bar{F}_\theta(\tau_{j-1})-\bar{F}_\theta(\tau_j)]\right.$$

$$\left.-\beta\,\Gamma\left(1+\frac{1}{\delta}\right)\right).\#(39)$$

It follows from "Eq. (39)" that

$$E\left\{C_{\widehat{\beta},\delta}(\tau^{\mathrm{unb}}(\gamma))\right\}=c_2E\left\{\vartheta\sum_{j=0}^{\infty}\bar{F}_{\widehat{\beta},\delta}(\tau_j^{\mathrm{unb}})\right.$$

$$\left.+\sum_{j=1}^{\infty}\tau_j^{\mathrm{unb}}[\bar{F}_\theta(\tau_{j-1}^{\mathrm{unb}})-\bar{F}_\theta(\tau_j^{\mathrm{unb}})]-\frac{\beta}{\widehat{\beta}}\widehat{\beta}\,\Gamma\left(1+\frac{1}{\delta}\right)\right\}$$

$$=c_2\left(\vartheta\sum_{j=0}^{\infty}E\left\{\bar{F}_\theta(\tau_j^{\mathrm{unb}})\right\}\right.$$

$$+\sum_{j=1}^{\infty}[r(\gamma^{-j/r}-1)]^{1/\delta}\widehat{\beta}E\left\{\bar{F}_\theta(\tau_{j-1}^{\mathrm{unb}})-\bar{F}_\theta(\tau_j^{\mathrm{unb}})\right\}$$

$$\left.-E\left\{V^{-1}\right\}\widehat{\beta}\,\Gamma\left(1+\frac{1}{\delta}\right)\right)=c_2\left(\vartheta\sum_{j=0}^{\infty}\gamma^j\right.$$

$$\left.+\sum_{j=1}^{\infty}[r(\gamma^{-j/r}-1)]^{1/\delta}\widehat{\beta}[\gamma^{j-1}-\gamma^j]-\frac{r\widehat{\beta}}{r-1}\Gamma\left(1+\frac{1}{\delta}\right)\right)$$

$$=c_2\begin{pmatrix}\vartheta/(1-\gamma)+r^{1/\delta}\widehat{\beta}(1/\gamma-1)\\ \times\sum_{j=1}^{\infty}\gamma^j[\gamma^{-j/r}-1]^{1/\delta}-\frac{r\widehat{\beta}}{r-1}\Gamma\left(1+\frac{1}{\delta}\right)\end{pmatrix}.\#(40)$$

Corollary 2.1. If δ=1, then

$$E\left\{C_{\widehat{\beta},\delta}(\tau^{\mathrm{unb}}(\gamma))\right\}$$

$$=c_2\left(\frac{\vartheta}{1-\gamma}+r\widehat{\beta}\frac{\gamma^{-1/r}-1}{1-\gamma^{1-1/r}}-\frac{r\widehat{\beta}}{r-1}\right)\#(41)$$

This ends the proof.

11. Minimization of the Total Expected Cost per Inspection Cycle via the Reliability Index γ

It follows from "Eq. (40)" that the fatigue reliability index γ minimizing the total estimated cost per inspection cycle is given by

$$\gamma^*=\arg\min_{0<\gamma<1}E\left\{C_{\widehat{\beta},\delta}(\tau^{\mathrm{unb}}(\gamma))\right\}$$

$$=\arg\min_{0<\gamma<1}\begin{pmatrix}\frac{\vartheta}{1-\gamma}+r^{1/\delta}\widehat{\beta}\left(\frac{1}{\gamma}-1\right)\\ \times\sum_{j=1}^{\infty}\gamma^j[\gamma^{-j/r}-1]^{1/\delta}\end{pmatrix}.\#(42)$$

If δ=1, It follows from "Eq. (41)" that the fatigue reliability index γ minimizing the total estimated cost per inspection cycle is given by

$$\gamma^*=\arg\min_{0<\gamma<1}E\left\{C_{\widehat{\beta},\delta}(\tau^{\mathrm{unb}}(\gamma))\right\}$$

$$=\arg\min_{0<\gamma<1}\left(\frac{\vartheta}{1-\gamma}+r\widehat{\beta}\frac{\gamma^{-1/r}-1}{1-\gamma^{1-1/r}}.\right)\#(43)$$

12. Index of Improvement Percentage in Effectiveness of Inspection Strategy

The index of improvement percentage in effectiveness of the optimal inspection strategy (with $\gamma=\gamma^*$) as compared with the standard inspection strategy (with $\gamma=\gamma_{\mathrm{st}}$) is given by

$$I_{\mathrm{imp.per}}(\gamma^*,\gamma_{\mathrm{st}})$$

$$=\frac{E\left\{C_{\widehat{\beta},\delta}(\tau(\gamma_{\mathrm{st}}))\right\}-E\left\{C_{\widehat{\beta},\delta}(\tau(\gamma^*))\right\}}{E\left\{C_{\widehat{\beta},\delta}(\tau(\gamma_{\mathrm{st}}))\right\}}100\%.\#(44)$$

13. Numerical Example 3

Let $X_1, \ldots, X_n$ be the random sample of the past independent observations of time to crack initiation (when a technically detectable crack is present) which follow the two-parameter Weibull distribution "Eq. (1)" , where $r = n = 2$, the shape parameter δ=1 and the scale parameter β is unknown. In this case, $\widehat{\beta}=2000$ hours. In order to construct the reliability-based inspection strategy for a new fatigued structure of the same type, the unbiased reliability-based inspection strategy "Eq. (30)" will be used. Let us assume that each inspection costs c_1=1 (in terms of money) and the cost of leaving an undetected failure (fatigue crack) is c_2=15 (in terms of money) per unit time.

It follows from "Eq. (43)" that

$$\gamma^*=\arg\min_{0<\gamma<1}\left(\frac{\vartheta}{1-\gamma}+r\widehat{\beta}\bullet\frac{\gamma^{-1/r}-1}{1-\gamma^{1-1/r}}\right)$$

$$=0.994251.\#(45)$$

It follows from "Eq. (41)" and "Eq. (45)" that

$$E\left\{C_{\widehat{\beta},\delta}(\tau^{\text{unb}}(\gamma^{*}))\right\}$$

$$=c_2\left(\frac{\vartheta}{1-\gamma^{*}}+r\widehat{\beta}\frac{\gamma^{*-1/r}-1}{1-\gamma^{*1-1/r}}-\frac{r\widehat{\beta}}{r-1}\right)$$

$$=347.1605.\#46$$

If the standard fatigue reliability index $\gamma_{\text{st}}=0.95$ is used, then

$$E\left\{C_{\widehat{\beta},\delta}(\tau^{\text{unb}}(\gamma_{\text{st}}))\right\}$$

$$=c_2\left(\frac{\vartheta}{1-\gamma_{st}}+r\widehat{\beta}\frac{\gamma_{\text{st}}^{-1/r}-1}{1-\gamma_{\text{st}}^{1-1/r}}-\frac{r\widehat{\beta}}{r-1}\right)$$

$$=1578.701.\#(47)$$

It follows from "Eq. (44)" that the index of improvement percentage in effectiveness of the unbiased reliability-based inspection strategy (with $\gamma=\gamma^{*}=0.994251$) as compared with the unbiased reliability-based inspection strategy (with $\gamma=\gamma_{\text{st}}=0.95$) is given by

$$I_{\text{imp.per}}(\gamma^{*},\gamma_{\text{st}})$$

$$=\left(1-\frac{E\left\{C_{\widehat{\beta}_{\bullet},\delta}(\tau^{\text{unb}}(\gamma^{*}))\right\}}{E\left\{C_{\widehat{\beta}_{\bullet},\delta}(\tau^{\text{unb}}(\gamma_{\text{st}}))\right\}}\right)100\%=78\%.\#(48)$$

Fig. 7 depicts the relationship between $E\left\{C_{\widehat{\beta},\delta}(\tau^{\text{unb}}(\gamma^{*}))\right\}$ and γ.

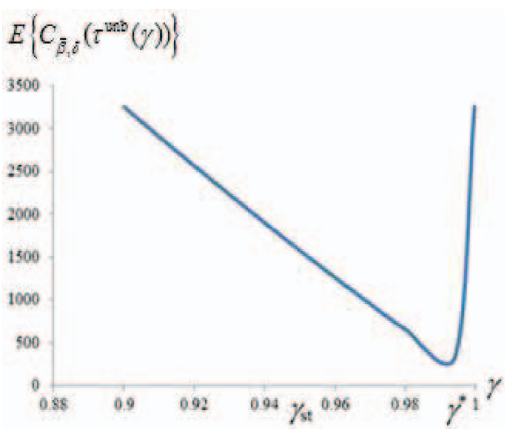

Fig. 7. Variation of $E\left\{C_{\widehat{\beta},\delta}(\tau^{\text{unb}}(\gamma))\right\}$ as a function of γ.

14. Unbiased Reliability-Based Inspection Strategy when Both the Scale Parameter β and the Shape Parameter δ Are Unknown

In this case, the unbiased reliability-based inspection strategy is determined as follows.

Theorem 3. Let $X_1 \le \ldots \le X_r$ be the first r ordered observations from a past random sample of size n of time to crack initiation (when a technically detectable crack is present) from the fatigued structures (components) of the same type, which follow the two-parameter Weibull distribution "Eq. (1)" where both the scale parameter β and the shape parameter δ are unknown. Then the unbiased reliability-based inspection strategy for a new fatigued structure (component) of the same type is given by

$$\tau_j^{\text{unb}}=\arg\left(\frac{1}{\vartheta(\mathbf{z}^{(r)})}\int_0^{\infty}v_2^{r-2}\prod_{i=1}^{r}z_i^{v_2}\right.$$

$$\left.\times\left[\sum_{i=1}^{r}z_i^{v_2}+(n-r)z_r^{v_2}+\left[\left(\tau_j/\widehat{\beta}\right)^{\widehat{\delta}}\right]^{v_2}\right]^{-r}dv_2=\gamma^{j}\right)$$

$$j=1,\ 2,\ 3,\ \ldots.\#(49)$$

Proof. The proof is similar to that of Theorem 1 and so it is omitted here.

Theorem 4. If (under conditions of Theorem 3) the unbiased reliability-based inspection strategy"Eq. (49)" is used, where each inspection costs c_1 and the cost of leaving an undetected failure (fatigue crack) is c_2 per unit time, then the fatigue reliability index γ minimizing the total expected cost per inspection cycle is given by

$$\gamma^{*}=\arg\min_{0<\gamma<1}E\left\{C_{\widehat{\theta}}(\tau^{\text{unb}}(\gamma))\right\}$$

$$=\arg\min_{0<\gamma<1}\left(\frac{\vartheta}{1-\gamma}+\frac{1-\gamma}{\gamma}\sum_{j=1}^{\infty}\tau_j^{\text{unb}}\gamma^{j}\right),\#(50)$$

where

$$E\left\{C_{\widehat{\theta}}(\tau^{\text{unb}}(\gamma))\right\}=c_2\left[\frac{\vartheta}{1-\gamma}+\frac{1-\gamma}{\gamma}\sum_{j=1}^{\infty}\tau_j^{\text{unb}}\gamma^{j}\right.$$

$$-\frac{1}{\vartheta(\mathbf{z}^{(r)})}\int_0^{\infty}\int_0^{\infty}v_2^{r-2}\prod_{i=1}^{r}z_i^{v_2}\left(\sum_{i=1}^{r}z_i^{v_2}\right.$$

$$\left.\left.+(n-r)z_r^{v_2}+\left[\left(x/\widehat{\beta}\right)^{\widehat{\delta}}\right]^{v_2}\right)^{-r}dv_2dx\right]\#(51)$$

Proof. The proof is similar to that of Theorem 2 and so it is omitted here.

15. Conclusion

In this paper, we discuss the optimal inspection strategy for the fatigue crack initiation period to prevent the reliability degradation due to fatigue damage and to minimize the expected total life cost. We propose a new technique of cost-effective planning reliability-based inspections of fatigued structures in the crack initiation period to derive the optimal inspection strategy easily. It is assumed that the fatigue component to be inspected is only one, which is immediately replaced by a virgin one if a crack is detected by the inspection. We illustrate the method of in-service inspection planning for the crack initiation period in the extreme-value and Weibull lifetime distribution cases. Applications to other distributions could follow directly.

The results obtained in this work can be used to solve the service problems of the following important engineering structures: (1) Transportation Systems and Vehicles – aircraft, space vehicles, trains, ships; (2) Civil Structures − bridges, dams, tunnels; (3) Power Generation – nuclear, fossil fuel and hydroelectric plants; (4) High-Value Manufactured Products − launch systems, satellites, semiconductor and electronic equipment; (5) Industrial Equipment − oil and gas exploration, production and processing equipment, chemical process facilities, pulp and paper.

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