Misspecification Analysis of a Gamma- with an Inverse Gaussian-Based Degradation Model in the presence of Measurement Error

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Gamma- and inverse Gaussian-based degradation models are natural choices for modelling monotonic degradation phenomena. Although not equivalent, in many applications these models are treated to each other. This situation makes the model misspecification problem interesting and important, especially when data are affected by measurement error, because from this kind of data it is not possible to check whether the selected model is able to fit the hidden degradation process or not. Motivated by the above considerations, in the paper we have carried out a small (i.e., very preliminary) Monte Carlo study to evaluate the effect produced by a misspecification of a gamma-with an inverse Gaussian-based perturbed degradation process. The study is performed considering as reference model a perturbed Gamma process recently proposed in the literature. The competing model is new and has been constructed with the aim of facilitating the misspecification study. In fact, it is an inverse Gaussian-based perturbed degradation model that share the same parameters and the same error term of the reference model. By virtue of the adopted setup, the mean and variance functions of the considered hidden Inverse Gaussian and Gamma processes have identical functional forms. The measurement error is modelled by a 3-parameter inverse gamma random variable, which depends in stochastic sense on the hidden degradation level. Model parameters are estimated by adopting the maximum likelihood method, via a sequential Monte Carlo approach. The fitting ability of the considered competing models is evaluated by using the Akaike information criterion. The effect of the misspecification is highlighted on the maximum likelihood estimate of mean remaining useful life. The impact of the presence of measurement error on the severity of the misspecification problem is also evaluated by comparing the obtained results with those attained by performing the same misspecification analysis in the absence of measurement error.

Keywords: Gamma process, inverse Gaussian process, degradation, measurement error, sequential Monte Carlo method, particle filter.

1. Introduction

Degradation data affected by measurement errors are often encountered in practice. In these situations, because in many reliability or maintenance applications it is important to study it, the hidden degradation process should be estimated by adopting models that are able to account for the presence of measurement error. When the degradation phenomenon of interest is known to be intrinsically monotone increasing, Gamma- and inverse Gaussian-based degradation models are natural choices for this kind of analyses (e.g., see Giorgio et al. (2019) and Hao et al. (2019)). Although not equivalent, in many applications these models are treated to each other. This situation makes the model misspecification problem interesting and important (e.g., see Zhang and Revie (2016) and Tseng and Yao (2017)), especially because data affected by errors do not allow checking whether the selected model is able to fit the hidden degradation process or not.

Motivated by the above considerations, in this paper a small Monte Carlo study is carried out to investigate the effect of misspecifying a gamma-with an inverse Gaussian-based degradation process. Measurement error is modelled by a 3-parameter inverse gamma random variable (r.v.) that depends (in stochastic sense) on the hidden degradation level. This modelling solution, that is...
motivated in Giorgio et al. (2019), distinguishes
the considered models from other perturbed
degradation models suggested in the literature,
where the error is assumed to be normally
distributed and/or stochastically independent
of the hidden degradation process (e.g., Le Son et al.
(2016), Bordes et al. (2016), and Hao et al.
(2019)). Model parameters are estimated by using
the maximum likelihood method, via a sequential
Monte Carlo approach. The fitting ability of the
considered competing perturbed models is
evaluated by using the Akaike information
criterion (see, Akaike (1974)). The effect of the
misspecification is highlighted on the maximum
likelihood estimates (MLEs) of the mean
remaining useful life (MRUL).

2. Gamma- and Inverse Gaussian-based
perturned degradation processes

A perturbed/noisy degradation process
\( \{Z(t), t \geq 0\} \) is a degradation model where the
actual degradation level \( W(t) \) cannot be observed
(i.e., it is hidden) due to the presence of a
perturbing term \( \varepsilon(t) \), that here is intended as a
measurement error. A perturbed process is usually
formulated as in Eq. (1):

\[
Z(t) = W(t) + \varepsilon(t) \tag{1}
\]

To fully define this kind of process it is necessary
to give a stochastic description of the hidden
process, error term, and relationships thereof.
The perturbed processes compared in this paper
are constructed by assuming that:

(i) As in Giorgio et al. (2019), the error term
\( \varepsilon(t) \) is a r.v. that depends in stochastic
sense on the actual degradation level
\( W(t) \). In fact, the conditional probability
density function (pdf) of \( \varepsilon(t) \), given
\( W(t) = w_t \) is formulated as:

\[
f_{\varepsilon(t)|W(t)}(\varepsilon|w) = \frac{[\alpha(w)]^{\beta(w)}(\varepsilon + w)^{-\beta(w) - 1}}{\Gamma[\beta(w)]} e^{-\frac{\alpha(w)}{\varepsilon + w}}, \tag{2}
\]

where:

\[
\beta(w) = \gamma w^{2-\gamma} + 2, \quad \gamma > 0, \quad -\infty < \nu < \infty
\]

and

\[
\alpha(w) = [\beta(w) - 1]w
\]

Hence, in particular, based on the pdf in
Eq. (2), the measurement error has conditional mean equal to zero:

\[
E[\varepsilon(t)|W(t) = w] = \frac{\alpha(w)}{\beta(w) - 1} - w = 0
\]

and conditional variance equal to:

\[
V[\varepsilon(t)|W(t) = w] = \frac{\alpha(w)^2}{\beta(w) - 2} = \frac{w^\nu}{\gamma}
\]

From Eq. (1), this assumption also implies that, the perturbed measurement \( Z(t) = W(t) + \varepsilon(t) \),
given \( W(t) = w \), is conditionally distributed as an inverse
gamma r.v., with pdf:

\[
f_{Z(t)|W(t)}(z|w) = \frac{[\alpha(w)]^{\beta(w)}z^{-\beta(w) - 1}}{\Gamma[\beta(w)]} e^{-\frac{\alpha(w)}{z}}, z \geq 0,
\]

conditional mean:

\[
E[Z(t)|W(t) = w] = E[\varepsilon(t)|W(t)] + w = w \tag{4}
\]

and conditional variance:

\[
V[Z(t)|W(t) = w] = V[\varepsilon(t)|W(t)] = \frac{w^\nu}{\gamma} \tag{5}
\]

(ii) For any \( n > 1 \), any \( j = 1, ..., n \), and any
set of measurement times \( t_1, ..., t_n \) the error term \( \varepsilon(t) \) given \( W(t) = w_f \) is conditionally
independent both on \( \varepsilon(t_1), ..., \varepsilon(t_{j-1}), \)
\( \varepsilon(t_{j+1}), ..., \varepsilon(t_n) \) and
\( W(t_1), ..., W(t_{j-1}), W(t_{j+1}), ..., W(t_n) \). Thus, equivalently, the
perturbed/noisy observation \( Z(t) \) given
\( W(t) = w_f \) is conditionally independent
both on \( Z(t_1), ..., Z(t_{j-1}), Z(t_{j+1}), ..., Z(t_n) \) and
\( W(t_1), ..., W(t_{j-1}), W(t_{j+1}), ..., W(t_n) \).

(iii) The hidden degradation process
\( \{W(t), t \geq 0\} \) is described by using two
different modelling solutions, both with
independent increments: I) a gamma
process and II) an inverse Gaussian
process. In the first case, the degradation
the mean function of total mean and total variance, respectively. In fact, Eq. 5, Eq. 8, and Eq. 9, by using the laws of inverse Gaussian degradation processes have obtained under this assumption is the one suggested in Giorgio et al. (2019).

In the second case, the degradation increments \( \Delta W(t_1, t_2) = W(t_2) - W(t_1) \) is an inverse Gaussian r.v., whose pdf, in this paper, is expressed as:

\[
f_{\Delta W(t_1, t_2)}(\delta_{1,2}) = \frac{\eta(t_1, t_2)}{\sqrt{2\pi\theta^{-1}}} e^{-\frac{[\delta_{1,2} - \theta\eta(t_1, t_2)]^2}{2\theta^3}}.
\]

where \( \theta \) is the scale parameter, \( \Gamma(\cdot) \) is the complete gamma function, \( \eta(t) \) is the shape function (usually also referred to as age function), and \( \eta(t_1, t_2) = \eta(t_2) - \eta(t_1) \). The perturbed gamma model obtained under this assumption is the one suggested in Giorgio et al. (2019).

Under these assumptions, the perturbed gamma and inverse Gaussian degradation processes have an identical mean function and a similar variance function, that can be easily formulated from Eq. 4, Eq. 5, Eq. 8, and Eq. 9, by using the laws of total mean and total variance, respectively. In fact, the mean function of \( Z(t) \) is equal to:

\[
E[Z(t)] = E[E[Z(t)]W(t)]
\]

and its variance is:

\[
V[Z(t)] = V[E[Z(t)]W(t)] + E[V[Z(t)]W(t)]
\]

where the parameters are denoted by using the same symbols adopted in Eq. 6. This special parameterization allows the mean and variance of the degradation increments of the considered gamma and Inverse Gaussian (hidden) processes to share exactly the same functional forms:

\[
E[\Delta W(t_1, t_2)] = \theta\eta(t_1, t_2) \quad (8)
\]

\[
V[\Delta W(t_1, t_2)] = \theta^2\eta(t_1, t_2) \quad (9)
\]

a feature that facilitates the misspecification analysis.

It is assumed that \( \eta(t) = 0 \) and \( W(0) = 0 \). Obviously, to fully define the considered hidden degradation models it is necessary to specify the functional form of the age function \( \eta(t) \).

Under these assumptions, the perturbed gamma and inverse Gaussian degradation processes have an identical mean function and a similar variance function, that can be easily formulated from Eq. 4, Eq. 5, Eq. 8, and Eq. 9, by using the laws of total mean and total variance, respectively. In fact, the mean function of \( Z(t) \) is equal to:

\[
E[Z(t)] = E[E[Z(t)]W(t)]
\]

and its variance is:

\[
V[Z(t)] = V[E[Z(t)]W(t)] + E[V[Z(t)]W(t)]
\]

whereas, under the perturbed inverse Gaussian it can be computed by solving the integral:

\[
E[(W(t))^\gamma] = \frac{\theta^\gamma \Gamma[\eta(t) + \nu]}{\Gamma[\eta(t)]}
\]

that, in general, does not allow for an elementary closed form solution and does not coincide with the fractal moment of \( W(t) \) given in Eq. (12). Nonetheless, when \( \nu = 2 \), given that the considered gamma and the inverse Gaussian (hidden) processes have the same mean, variance, and (hence) second moment:

\[
E[(W(t))^2] = \theta^2\eta(t)[\eta(t) + 1]
\]

under both perturbed models it will be:

\[
V[Z(t)] = \theta^2[2\eta(t) + [\eta(t)]^2].
\]

It is worth to mention that, when \( \nu = 2 \) the conditional standard deviation of the error term \( \varepsilon(t) \), given \( W(t) = w \), is a linear function of the actual (hidden) degradation level \( W(t) \):

\[
\sqrt{V[\varepsilon(t)|W(t) = w]} = \frac{w}{\sqrt{\gamma}}
\]

3. Likelihood function and Cdf of RUL

Let us suppose that the degradation level of \( m \) units, operating under identical conditions, is measured at ages \( t_i \) (\( i = 1, \ldots, m \) and \( j = 1, \ldots, n_i \)) and assume that the degradation measurements are contaminated by random errors. Let \( z_{ij} \) denote the value of the perturbed degradation level \( Z(t_i) \) at \( t_i \). Then, under the
perturbed processes presented in Section 2, the likelihood function \( L(\xi; z) \) of the available data can be expressed as:

\[
L(\xi; z) = \prod_{i=1}^{m} \prod_{j=1}^{n_i} f_{z_{ij}}(z_{ij} | z_{i,j-1})
\]

(12)

where, \( Z_i = \{z_{i1}, ..., Z_i\} \) denotes the set of perturbed measurements made on the unit \( i \) up to the time \( t_i \), \( z_i = \{z_{i1}, ..., z_{ij}\} \) denotes its realization, \( Z = \{z_{11}, ..., z_{m1}, ..., z_{mn}\} \) is the whole vector of noisy measurements, \( \xi \) denotes the vector of model parameters, \( t_0 = 0 \), and \( Z_0 \) is an empty set. A degrading unit is conventionally considered failed when its degradation level exceeds an assigned threshold value \( w_M \). On the basis of this definition of failure, the remaining useful life \( RUL(t) \) is determined as the (non-negative) time elapsing from \( t \) to the first passage time of the hidden degradation process to \( w_M \). In this paper, it is assumed that \( RUL(t) = 0 \) if the process has already passed the threshold before \( t \) and that failures are not self-announcing. Consequently, considered that both the gamma and the inverse Gaussian processes are monotonic increasing, the complementary Cdf of \( RUL(t) \) is formulated as the conditional probability that, given all the perturbed measurements gathered up to time \( t \), the actual degradation level at time \( t + \tau \) does not exceed the threshold limit:

\[
\begin{align*}
\bar{F}_{RUL(t)}(\tau | z_t) &= 1 - P(RUL(t) \leq \tau | z_t) \\
&= \int_{0}^{\tau} \int f_{W_{t+i}|W_t}(w_M | w_t) f_{W_t|z_t}(w_t | z_t) \, dw_t \\
&= F_{W_{t+i}|z_t}(w_M | z_t)
\end{align*}
\]

(13)

where \( W_{t+i} \) denotes the actual degradation level at \( t + \tau \), \( W_t \) is the actual degradation level at \( t \), \( Z = \{Z_i; 0 \leq i \leq n, t_i < t\} \) is the set of all the measurement collected up to \( t \), and \( z_t \) is its realization. Based on the complementary Cdf in Eq. (13) the mean of \( RUL(t) \) can be computed as:

\[
M_{RUL}(t) = E[RUL(t)] = \int_{0}^{\infty} \bar{F}_{RUL(t)}(\tau | z_t) \, d\tau
\]

(14)

4. The particle filter

The particle filter method (e.g., see Simon (2006)) allows computing the pdfs in Eq. (15), Eq. (16), and Eq. (17), by using a sequential Monte Carlo approach, once the value of the parameter vector \( \xi \) is assigned (that is, model parameters should be
The particles from the r.v. first set of particles (i.e., $W_i$), whose pdf is given in Eq. (14). The elements of the random sample are called particles. Denote these particles by $w_{i,1},\ldots,w_{i,N}$;

(ii) Update step: For $k = 1, \ldots, N$, evaluate the normalized importance weights as:

$$kq_{i,j} = \frac{f_{z_i|W_i}(z_{i,j} | k w_{i,j-})}{\sum_{k=1}^{N} f_{z_i|W_i}(z_{i,j} | k w_{i,j-})}$$

where $f_{Z_i|W}( \cdot | \cdot )$ is the pdf in Eq. (3). Hence, resample new particles from the set $w_{i,1},\ldots,w_{i,N}$ according to the importance weights. Rename the new particle as $w_{i,1},\ldots,w_{i,N}$.

The algorithm is initialized by setting $j = 1$. The first set of particles (i.e., $w_{i,1},\ldots,w_{i,N}$) is drawn from the r.v. $W_i$, whose pdf is either the one in Eq. (6) (under the gamma process) or the one in Eq. (7) (under the inverse Gaussian process), with $t_i=0$, $t_i=t_1$. For $j > 1$, the particles $w_{i,j},\ldots,w_{i,N}$ are generated by using the formula:

$$k w_{i,j-} = k w_{i,j-1} + k \Delta W_{i,j}$$

where the term $k \Delta W_{i,j}$ is drawn from the r.v. $\Delta W_{i,j}=W_{i,j}-W_{i,j-1}$, whose pdf is either the one in Eq. (6) (under the gamma process) or the one in Eq. (7) (under the inverse Gaussian process), with $t_i=t_{i-1}$, $t_i=t_i$. The procedure is repeated while $j \leq n$.

The particles $w_{i,1},\ldots,w_{i,N}$ can be viewed as realizations of $W_i|z_{i,1}$. These particles can be used to compute, for any $j$, the pdf of $Z_i|z_{i,1}$, that is needed to evaluate the likelihood function in Eq. (12). In particular, the value of the pdf of $Z_i|z_{i,1}$ at $z_{i,j}$ can be computed as:

$$f_{z_i|W_{i,j}}(z_{i,j} | k w_{i,j-}) = \frac{1}{N}$$

The same particle filter method also permits to generate pseudo-random realizations of $w_{i,1},z_{i,j}$ that can be used to estimate the residual reliability in Eq. (13) by evaluating the empirical Cdf of $W_{i,t}|z_{i,j}$ at $w_{i,t}$, once the set of parameters $\tilde{\theta}$ that maximize the likelihood function has been obtained. For further details about this algorithm see Giorgio et al. (2019).

4. Misspecification analysis
In order to perform the misspecification analysis, we have generated two sets of degradation measurements (hereinafter referred to as datasets $\mathcal{A}$ and $\mathcal{B}$) under two perturbed gamma degradation processes, with equal mean and different variance to mean ratio, that (given the mean) increases with $\theta$. The error term has been calibrated by using always the same setup. The parameter settings used to generate the datasets are reported in table 1. Data are in tables 2 and 3.

<table>
<thead>
<tr>
<th>Dataset</th>
<th>$a$</th>
<th>$b$</th>
<th>$\theta$</th>
<th>$\gamma$</th>
<th>$\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{A}$</td>
<td>0.003</td>
<td>0.85</td>
<td>0.0008</td>
<td>60</td>
<td>3</td>
</tr>
<tr>
<td>$\mathcal{B}$</td>
<td>1.5</td>
<td>0.85</td>
<td>0.1575</td>
<td>60</td>
<td>3</td>
</tr>
</tbody>
</table>

where $1.5 = 0.003 \times 500$ and $0.1575 = 0.0008 \times 500^{0.85}$. Each dataset consists of 80 measurements. Specifically, there are $m=10$ units and $n=8$ measurements for each unit. Measurement times are the same for all the units (i.e., it is $t_{i,j} = t_i \forall i, \ i = 1,2,\ldots,10$). Besides the perturbed measurements, the tables 2 and 3 also report (in parentheses) the hidden degradation levels at each measurement epoch, which in the case of the considered Monte Carlo study are known.

The settings in table 1 have been defined considering that risk of incurring in a misspecification mainly depends on the values of the shape parameters of the increments of the gamma hidden process between successive measurement times (i.e., $\eta(t_1), \eta(t_3,t_2),\ldots, \eta(t_7,t_6)$). In fact, under the considered parameterization, both the gamma and the inverse Gaussian random variables, tend to a Gaussian r.v. (i.e., the same Gaussian r.v.) as $\eta(\cdot)$ tends to infinity. Hence, the larger $\eta(\cdot)$ the higher the risk of incurring in a misspecification. Based on these premises, the risk of a misspecification is expected to be severe in the case of the dataset $\mathcal{A}$ and relatively mild in the case of the dataset $\mathcal{B}$. In fact, for example, under the settings $\mathcal{A}$ and $\mathcal{B}$, in the absence of measurement error, the Akaike information criterion leads to misspecify the
gamma process as an inverse Gaussian process with probability 0.46 and 0.0068, respectively.

Table 2. Dataset $\mathcal{A}$

<table>
<thead>
<tr>
<th>Unit #</th>
<th>$t = 2$</th>
<th>$t = 4$</th>
<th>$t = 6$</th>
<th>$t = 8$</th>
<th>$t = 10$</th>
<th>$t = 12$</th>
<th>$t = 14$</th>
<th>$t = 16$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.24 (0.23)</td>
<td>0.36 (0.37)</td>
<td>0.57 (0.52)</td>
<td>0.56 (0.66)</td>
<td>0.71 (0.80)</td>
<td>1.13 (0.92)</td>
<td>0.93 (1.06)</td>
<td>1.16 (1.18)</td>
</tr>
<tr>
<td>2</td>
<td>0.18 (0.21)</td>
<td>0.40 (0.39)</td>
<td>0.53 (0.54)</td>
<td>0.71 (0.70)</td>
<td>0.85 (0.85)</td>
<td>1.00 (0.98)</td>
<td>1.03 (1.10)</td>
<td>1.14 (1.22)</td>
</tr>
<tr>
<td>3</td>
<td>0.18 (0.18)</td>
<td>0.33 (0.36)</td>
<td>0.48 (0.51)</td>
<td>0.67 (0.65)</td>
<td>0.94 (0.79)</td>
<td>1.12 (0.92)</td>
<td>0.92 (1.04)</td>
<td>1.02 (1.16)</td>
</tr>
<tr>
<td>4</td>
<td>0.22 (0.21)</td>
<td>0.35 (0.36)</td>
<td>0.56 (0.52)</td>
<td>0.63 (0.65)</td>
<td>0.65 (0.79)</td>
<td>1.07 (0.93)</td>
<td>1.14 (1.05)</td>
<td>1.09 (1.17)</td>
</tr>
<tr>
<td>5</td>
<td>0.23 (0.24)</td>
<td>0.39 (0.39)</td>
<td>0.56 (0.55)</td>
<td>0.68 (0.72)</td>
<td>0.83 (0.84)</td>
<td>0.96 (0.97)</td>
<td>0.92 (1.08)</td>
<td>1.07 (1.22)</td>
</tr>
<tr>
<td>6</td>
<td>0.28 (0.24)</td>
<td>0.40 (0.39)</td>
<td>0.46 (0.54)</td>
<td>0.78 (0.69)</td>
<td>0.82 (0.83)</td>
<td>0.90 (0.95)</td>
<td>1.06 (1.07)</td>
<td>1.05 (1.20)</td>
</tr>
<tr>
<td>7</td>
<td>0.20 (0.20)</td>
<td>0.20 (0.20)</td>
<td>0.35 (0.36)</td>
<td>0.69 (0.64)</td>
<td>0.76 (0.81)</td>
<td>0.81 (0.93)</td>
<td>0.92 (1.06)</td>
<td>1.19 (1.18)</td>
</tr>
<tr>
<td>8</td>
<td>0.19 (0.20)</td>
<td>0.20 (0.20)</td>
<td>0.35 (0.38)</td>
<td>0.69 (0.67)</td>
<td>0.82 (0.81)</td>
<td>0.90 (0.94)</td>
<td>1.06 (1.07)</td>
<td>1.61 (1.20)</td>
</tr>
<tr>
<td>9</td>
<td>0.22 (0.22)</td>
<td>0.36 (0.37)</td>
<td>0.54 (0.53)</td>
<td>0.93 (0.67)</td>
<td>1.02 (0.79)</td>
<td>1.05 (0.90)</td>
<td>1.45 (1.04)</td>
<td>1.40 (1.18)</td>
</tr>
<tr>
<td>10</td>
<td>0.21 (0.22)</td>
<td>0.34 (0.36)</td>
<td>0.53 (0.52)</td>
<td>0.58 (0.66)</td>
<td>0.75 (0.78)</td>
<td>0.89 (0.90)</td>
<td>1.37 (1.03)</td>
<td>1.04 (1.15)</td>
</tr>
</tbody>
</table>

Table 3. Dataset $\mathcal{B}$

<table>
<thead>
<tr>
<th>Unit #</th>
<th>$t = 2$</th>
<th>$t = 4$</th>
<th>$t = 6$</th>
<th>$t = 8$</th>
<th>$t = 10$</th>
<th>$t = 12$</th>
<th>$t = 14$</th>
<th>$t = 16$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.05 (0.05)</td>
<td>0.44 (0.42)</td>
<td>0.54 (0.55)</td>
<td>0.55 (0.57)</td>
<td>0.69 (0.66)</td>
<td>0.82 (0.89)</td>
<td>1.08 (1.02)</td>
<td>1.13 (1.15)</td>
</tr>
<tr>
<td>2</td>
<td>0.08 (0.08)</td>
<td>0.42 (0.42)</td>
<td>0.53 (0.53)</td>
<td>0.65 (0.61)</td>
<td>0.84 (0.89)</td>
<td>1.64 (1.40)</td>
<td>1.73 (1.45)</td>
<td>1.44 (1.55)</td>
</tr>
<tr>
<td>3</td>
<td>0.07 (0.08)</td>
<td>0.13 (0.12)</td>
<td>0.13 (0.13)</td>
<td>0.20 (0.21)</td>
<td>0.41 (0.42)</td>
<td>0.51 (0.47)</td>
<td>0.57 (0.53)</td>
<td>0.62 (0.54)</td>
</tr>
<tr>
<td>4</td>
<td>0.05 (0.05)</td>
<td>0.17 (0.18)</td>
<td>0.62 (0.62)</td>
<td>0.67 (0.64)</td>
<td>0.65 (0.76)</td>
<td>0.86 (0.87)</td>
<td>1.16 (1.00)</td>
<td>0.86 (1.14)</td>
</tr>
<tr>
<td>5</td>
<td>0.01 (0.01)</td>
<td>0.08 (0.09)</td>
<td>0.33 (0.38)</td>
<td>0.59 (0.64)</td>
<td>0.68 (0.65)</td>
<td>0.77 (0.74)</td>
<td>1.23 (0.83)</td>
<td>0.99 (0.86)</td>
</tr>
<tr>
<td>6</td>
<td>0.13 (0.13)</td>
<td>0.22 (0.20)</td>
<td>0.31 (0.28)</td>
<td>0.41 (0.45)</td>
<td>0.75 (0.65)</td>
<td>0.75 (0.83)</td>
<td>1.03 (0.88)</td>
<td>1.16 (1.08)</td>
</tr>
<tr>
<td>7</td>
<td>0.19 (0.19)</td>
<td>0.28 (0.2500)</td>
<td>0.24 (0.2502)</td>
<td>0.49 (0.50)</td>
<td>0.67 (0.66)</td>
<td>0.69 (0.69)</td>
<td>0.85 (0.84)</td>
<td>0.89 (1.01)</td>
</tr>
<tr>
<td>8</td>
<td>0.02 (0.02)</td>
<td>0.14 (0.13)</td>
<td>0.51 (0.46)</td>
<td>0.55 (0.54)</td>
<td>1.28 (1.15)</td>
<td>1.30 (1.16)</td>
<td>0.98 (1.31)</td>
<td>2.45 (1.76)</td>
</tr>
<tr>
<td>9</td>
<td>0.09 (0.09)</td>
<td>0.33 (0.30)</td>
<td>0.60 (0.49)</td>
<td>0.54 (0.51)</td>
<td>0.51 (0.52)</td>
<td>0.66 (0.69)</td>
<td>0.76 (0.75)</td>
<td>0.97 (1.03)</td>
</tr>
<tr>
<td>10</td>
<td>0.38 (0.40)</td>
<td>0.91 (0.96)</td>
<td>1.20 (1.02)</td>
<td>1.26 (1.18)</td>
<td>1.28 (1.26)</td>
<td>1.52 (1.38)</td>
<td>1.49 (1.41)</td>
<td>1.38 (1.51)</td>
</tr>
</tbody>
</table>

Table 4. MLEs of the parameters and AIC index under the considered competing models, from dataset $\mathcal{A}$.

<table>
<thead>
<tr>
<th>MLEs</th>
<th>True</th>
<th>PG</th>
<th>PIG</th>
<th>G</th>
<th>IG</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>$3 \times 10^{-3}$</td>
<td>$3 \times 10^{-3}$</td>
<td>$3 \times 10^{-3}$</td>
<td>$3 \times 10^{-3}$</td>
<td>$3 \times 10^{-3}$</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.85</td>
<td>0.85</td>
<td>0.84</td>
<td>0.83</td>
<td>0.83</td>
</tr>
<tr>
<td>$\theta$</td>
<td>$8 \times 10^{-4}$</td>
<td>$9 \times 10^{-4}$</td>
<td>$1 \times 10^{-3}$</td>
<td>$1 \times 10^{-3}$</td>
<td>$1 \times 10^{-3}$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>60</td>
<td>60</td>
<td>60</td>
<td>60</td>
<td>60</td>
</tr>
<tr>
<td>AIC</td>
<td>-175.4</td>
<td>-175.5</td>
<td>-465.8</td>
<td>-466.1</td>
<td>-466.1</td>
</tr>
</tbody>
</table>

The MLEs of the parameters and the values of the AIC index reported in tables 4 indicate that, in the case of the dataset $\mathcal{A}$, the considered competing models can be deemed practically equivalent, both in the presence and in the absence of measurement error. In fact, for example, as shown in the figures 1 and 2, the estimated perturbed processes provide identical fits of the empirical estimates of $E[Z(t)]$, $E[W(t)]$, $V[Z(t)]$, $V[W(t)]$. The TRUE MRUL is computed under the gamma process setting the parameters to the values used to simulate the data. The MRUL values reported in the columns TRUE, G, and IG are obtained assuming that the actual degradation level at $t = 16$ is known. The tables 6 and 7 report the analogue results obtained from the dataset $\mathcal{B}$.
The results obtained from the dataset \( \mathcal{B} \), seem to give different indications. In fact, in this case, the MLEs and the AIC reported in table 6 (especially in the absence of measurement error) show that the gamma model outperforms the inverse Gaussian one in terms of fitting ability. Nonetheless, the estimates of the MRUL(16) reported in table 7, obtained under the perturbed models, are still close ones to the others and none of the models seems to outperform the other in terms of ability to estimate the MRUL.

### Table 6. MLEs of the parameters and AIC index obtained under the considered competing models from the dataset \( \mathcal{B} \).

<table>
<thead>
<tr>
<th></th>
<th>TRUE</th>
<th>PG</th>
<th>PIG</th>
<th>G</th>
<th>IG</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{a} )</td>
<td>1.5</td>
<td>1.74</td>
<td>2.59</td>
<td>1.84</td>
<td>15.3</td>
</tr>
<tr>
<td>( \hat{b} )</td>
<td>0.85</td>
<td>1.05</td>
<td>1.13</td>
<td>1.04</td>
<td>1.65</td>
</tr>
<tr>
<td>( \hat{\theta} )</td>
<td>0.1575</td>
<td>0.12</td>
<td>0.16</td>
<td>0.12</td>
<td>1.08</td>
</tr>
<tr>
<td>( \hat{\gamma} )</td>
<td>60</td>
<td>45.4</td>
<td>48.0</td>
<td>/</td>
<td>/</td>
</tr>
<tr>
<td>( \hat{\nu} )</td>
<td>3</td>
<td>2.93</td>
<td>2.34</td>
<td>/</td>
<td>/</td>
</tr>
<tr>
<td>AIC</td>
<td>170.3</td>
<td>172.7</td>
<td>-140</td>
<td>-3</td>
<td></td>
</tr>
</tbody>
</table>

### Table 7. TRUe value and MLEs of MRUL(16) obtained under the considered competing models from dataset \( \mathcal{B} \).

<table>
<thead>
<tr>
<th></th>
<th>TRUE</th>
<th>PG</th>
<th>PIG</th>
<th>G</th>
<th>IG</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{a} )</td>
<td>1.89</td>
<td>1.25</td>
<td>1.22</td>
<td>1.36</td>
<td>1.69</td>
</tr>
<tr>
<td>( \hat{b} )</td>
<td>0</td>
<td>4.10^{-4}</td>
<td>7.10^{-4}</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \hat{\theta} )</td>
<td>12.45</td>
<td>7.33</td>
<td>6.76</td>
<td>9.36</td>
<td>7.54</td>
</tr>
<tr>
<td>( \hat{\gamma} )</td>
<td>2.08</td>
<td>2.29</td>
<td>2.26</td>
<td>1.50</td>
<td>1.86</td>
</tr>
<tr>
<td>( \hat{\nu} )</td>
<td>6.58</td>
<td>1.56</td>
<td>1.57</td>
<td>4.97</td>
<td>4.83</td>
</tr>
<tr>
<td>( \hat{\omega} )</td>
<td>3.17</td>
<td>1.35</td>
<td>1.31</td>
<td>2.34</td>
<td>2.74</td>
</tr>
<tr>
<td>( \hat{\psi} )</td>
<td>4.37</td>
<td>3.76</td>
<td>3.61</td>
<td>3.27</td>
<td>3.55</td>
</tr>
<tr>
<td>( \hat{\phi} )</td>
<td>8</td>
<td>2.10^{-4}</td>
<td>4.10^{-4}</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \hat{\chi} )</td>
<td>9</td>
<td>3.70</td>
<td>3.56</td>
<td>3.00</td>
<td>3.33</td>
</tr>
<tr>
<td>( \hat{\tau} )</td>
<td>4.03</td>
<td>2.10^{-5}</td>
<td>2.10^{-5}</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Actually, figures 3 and 4 show that also the (outperformed) perturbed inverse Gaussian model fits the data adequately. We finally note that, the AIC values reported in the last row of table 6 also show that the presence of measurement error increases the risk of incurring in a wrong diagnosis. This result was expected because the measurement error may mask features of the degradation paths that can give precious indications about data generating process (e.g., the presence of very tiny degradation increments and/or skewness in the data).
Fig. 3. Empirical estimates of $V[Z(t)]$ and $V[E(t)]$ together with the MLEs of $V[Z(t)]$ and $V[W(t)]$ obtained from the dataset $B$.

Fig. 4. Empirical estimates of $E[Z(t)]$ and $E[W(t)]$ together with the MLEs of $E[Z(t)]$ obtained under the perturbed gamma (PG) and inverse Gaussian models (PIG) from the dataset $B$.

5. Conclusions

The point of the paper is investigating the model misspecification issue of a gamma- with an inverse Gaussian-based perturbed degradation process. The problem is faced by adopting as competing models a perturbed gamma process recently proposed in the literature and a new inverse Gaussian perturbed degradation model that share the same parameters and functional forms of the mean and variance functions. The proposed setup is adopted to perform a preliminary study based on two simulated datasets, generated under the considered gamma perturbed process.

Obtained results indicate that, as expected, the risk of a misspecification mainly depends on the values of the shape parameters of the increments of the hidden gamma process between successive measurement times and that the presence of measurement error increases the risk of incurring in a wrong diagnosis. Yet, based on the performed investigation, it seems not certain that the misspecification of a perturbed gamma with a perturbed inverse Gaussian process affects in a severe manner the estimates of the mean remaining useful life.

References


