Proceedings of the 32nd European Safety and Reliability Conference (ESREL 2022)
Edited by Maria Chiara Leva, Edoardo Patelli, Luca Podofillini, and Simon Wilson
 Published by Research Publishing, Singapore.
doi: 10.3850/978-981-18-5183-4_R10-02-189-cd

Optimizing Multiple Reinsurance Contracts

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An insurance contract implies that risk is ceded from ordinary policy holders to companies. Companies do the same thing between themselves, and this is known as reinsurance. The problem of determining reinsurance contracts which are optimal with respect to some reasonable criterion has been studied extensively. Different contract types are considered such as stop-loss contracts where the reinsurance company covers risk above a certain level, and insurance layer contracts where the reinsurance company covers risk within an interval. The contracts are then optimized with respect to some risk measure, such as value-at-risk or conditional tail expectation. In the present paper we investigate this problem further and show that the optimal solution depends on the tail hazard rates of the risk distributions. If the tail hazard rates are decreasing, which is the case for heavy tailed distributions like lognormal and pareto distributions, the optimal solution is balanced. That is, reinsurance contracts for identically distributed risks should be identical insurance layer contracts. However, if the tail hazard rate is increasing, which is the case for light tailed distributions like truncated normal distributions, the optimal solution is typically not balanced. Even for identically distributed risks, some contracts should be insurance layer contracts, while others should be stop-loss contracts. In the limiting case, where the hazard rate is constant, i.e., when the risks are exponentially distributed, we show that a balanced solution is optimal. We also present an efficient importance sampling method for estimating optimal contracts.

Keywords: optimal reinsurance, multivariate optimization, value-at-risk, importance sampling

1. Introduction

When a client buys an insurance contract, risk is ceded from the client to the insurance company. When the client is an insurance company, referred to as the *cedent*, this is known as *reinsurance*. Typically, reinsurance is applied either to very large single risks or to a portfolio of risks. Even the largest companies do this as a part of their diversification strategy, and financially the cedent may be just as strong as the reinsurer.

The problem of determining reinsurance contracts which are optimal with respect to some reasonable criterion has been studied extensively within actuarial science. Different contract types are considered such as *stop-loss contracts* where the reinsurer covers risk above a certain level, and *insurance layer contracts* where the reinsurer covers risk within an interval. The contracts can then be optimized with respect to some risk measure such as value-at-risk or conditional tail expectation. In the univariate case, i.e., when only a single risk is reinsured, solutions to several variations of the optimization problem can be found in Cheung et al. (2014). In particular, it is shown that if value-at-risk is used as risk measure, the optimal contract is known to be an insurance layer contract. Some other recent works in this area are Lu et al. (2013), Cong and Tan (2016), and Chi et al. (2017).

The topic of the present paper is the multivariate case where the cedent has multiple risks which cannot be bundled together into a portfolio. Instead the risks must be covered by separate reinsurance contracts. In this case the problem of finding optimal contracts is more difficult. More specifically, we consider the case where value-at-risk is used as risk measure. Since an insurance layer contract is known to be optimal with this risk measure, we focus on such contracts. The risks covered by the reinsurer are characterized by intervals. This means that for each contract we have two parameters corresponding to the bounds of these intervals. Thus, if m is the number of contracts, we have an optimization problem with a total $2m$ parameters. In principle, it is possible to find the optimal parameters numerically. However, since the computational order of the optimization problem typically grows exponentially in the number of parameters, the computational complexity of the optimization soon becomes very large.

In Huseby and Christensen (2020), it was shown that optimal solutions must satisfy certain conditions. These conditions simplify the optimization problem significantly. In the present paper, we apply these conditions and show that the optimization problem can be expressed as an optimization problem with just m variables subject to a single constraint. Expressed in terms of a transformed set of variables the constraint is approximately linear, and the optimal solution depends on the shape of the upper and lower contour sets of the expected reinsurance cost. When these sets are convex, optimal solutions can be found efficiently using Lagrange optimization. On the other hand, when the upper contour sets are convex, optimal solutions will be located at the boundary of the set of feasible contracts.

1.1. *Value-at-risk*

Since value-at-risk plays an important part in the present paper, we review some basic properties of this risk measure which will be needed later on. The risk measure will typically be defined relative to some random variable X. The cumulative distribution function of X is denoted by $F_X(x) = P(X \leq x)$. We also introduce the survival function $S_X(x) = P(X > x) = 1 - F_X(x)$.

The α-level value-at-risk associated with the risk X is given by $S_X^{-1}(\alpha)$ defined as follows:

$$S_X^{-1}(\alpha) = \inf\{x : P(X > x) \leq \alpha\}. \quad (1)$$

If S_X is strictly decreasing, it is easy to show that:

$$S_X^{-1}(\alpha) = r \text{ if and only if } P(X > r) \leq \alpha \leq P(X \geq r). \quad (2)$$

In particular, when S_X is strictly decreasing, the following holds true:

$$\text{If } P(X > r) = \alpha, \text{ then } S_X^{-1}(\alpha) = r. \quad (3)$$

It is well-known that the value-at-risk function has the property that for any strictly increasing continuous function ϕ we have:

$$S_{\phi(X)}^{-1}(\alpha) = \phi(S_X^{-1}(\alpha)) \quad (4)$$

2. Multivariate reinsurance contracts

In this section we consider the problem of optimizing insurance contracts in the multivariate case. Thus, we let $X_1, \ldots, X_m$ be m non-negative random variables representing the risks from m business lines[a]. To avoid technical issues we assume that $\boldsymbol{X} = (X_1, \ldots, X_m)$ is absolutely continuously distributed, and that S_{X_i} is strictly decreasing, $i = 1, \ldots, m$.

All risks are reinsured using *insurance layer* contracts. That is, for $i = 1, \ldots, m$ we introduce:

$$R_i(X_i) = \begin{cases} 0 & \text{for } X_i < a_i \\ X_i - a_i & \text{for } a_i \leq X_i \leq b_i \\ b_i - a_i & \text{for } X_i > b_i \end{cases} \quad (5)$$

where $a_i < b_i$ are positive constants. As a result, the retained risks covered by the cedent, denoted $I_i(X_i) = X_i - R_i(X_i)$, $i = 1, \ldots, m$, are:

$$I_i(X_i) = \begin{cases} X_i & \text{for } X_i < a_i \\ a_i & \text{for } a_i \leq X_i \leq b_i \\ X_i - (b_i - a_i) & \text{for } X_i > b_i \end{cases} \quad (6)$$

The price paid by the cedent for the ith contract is denoted by π_{X_i}, and is assumed to be of the form:

$$\pi_{X_i} = (1 + \theta)E[R_i(X_i)], \quad i = 1, \ldots, m.$$

where $\theta > 0$ represents the risk premiums charged by the reinsurance company for handling the risks[b]. The total risk covered by the cedent is then:

$$\sum_{i=1}^{m} I_i(X_i) + (1 + \theta) \sum_{i=1}^{m} E[R_i(X_i)].$$

By Eq. (4) the resulting total α-level value-at-risk, denoted V_α, is given by:

$$V_\alpha = S_{\sum_{i=1}^m I_i(X_i)}^{-1}(\alpha) + (1 + \theta) \sum_{i=1}^{m} E[R_i(X_i)].$$

[a]Note that the risks $X_1, \ldots, X_m$ are *not* assumed to be independent at this stage.

[b]For more sophisticated non-linear premium principles see Bühlmann (1980) and Furman and Zitikis (2007)

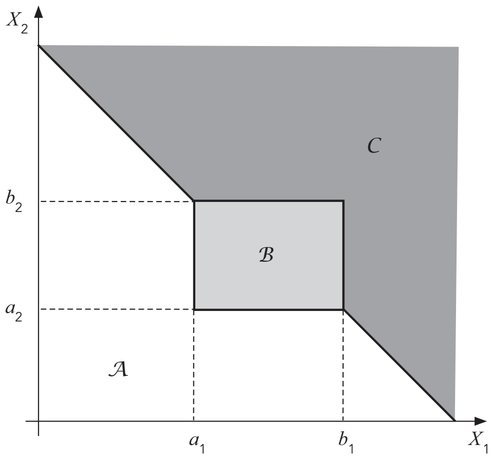

Fig. 1. The sets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$.

We observe that V_α consists of two terms. We refer to the first term,

$$S^{-1}_{\sum_{i=1}^m I_i(X_i)}(\alpha)$$

as the *retained risk term*, while:

$$\sum_{i=1}^m \pi_{X_i} = (1+\theta)\sum_{i=1}^m E[R_i(X_i)]$$

is referred to as the *premium term*. The main objective now is to find $a_1, b_1, \ldots, a_m, b_m$ so that V_α is minimized.

We then let $\boldsymbol{x} = (x_1, \ldots, x_m)$ and introduce the following sets which corresponds to different scenarios of the retained risk term:

$$\mathcal{A} = \{\boldsymbol{x} : \sum_{i=1}^m I_i(x_i) < \sum_{i=1}^m a_i\}, \tag{7}$$

$$\mathcal{B} = \{\boldsymbol{x} : \sum_{i=1}^m I_i(x_i) = \sum_{i=1}^m a_i\}, \tag{8}$$

$$\mathcal{C} = \{\boldsymbol{x} : \sum_{i=1}^m I_i(x_i) > \sum_{i=1}^m a_i\}. \tag{9}$$

These sets are illustrated for the case where $m = 2$ in Figure 1. Note that the subset $\mathcal{B}$ also includes the boundary of the rectangle as well as the borderline between $\mathcal{A}$ and $\mathcal{C}$. Moreover, since we have assumed that S_{X_i} are strictly decreasing for all i, it follows that $P(\boldsymbol{X} \in \mathcal{B} \cup \mathcal{C})$ and $P(\boldsymbol{X} \in \mathcal{C})$ are strictly decreasing in a_i for all i.

In Huseby and Christensen (2020) the following important result was proven[c]:

Theorem 2.1. *Assume that* $a_1^*, b_1^*, \ldots, a_m^*, b_m^*$ *are optimal contract parameter values, and that*

$$P(\bigcap_{i=1}^m X_i > a_i^*) \geq \alpha. \tag{10}$$

Then the following conditions must hold true:

$$a_i^* = S^{-1}_{X_i}(\frac{1}{1+\theta}), \quad i = 1, \ldots, m, \tag{11}$$

and:

$$P(\boldsymbol{X} \in \mathcal{C}) = \alpha. \tag{12}$$

If more information about the joint distribution of $X_1, \ldots, X_m$ is available, it is possible to simplify the condition Eq. (10). In particular, if $X_1, \ldots, X_m$ are *positively upper orthant dependent* (see Shaked (1982)), i.e.,

$$P(\bigcap_{i=1}^m X_i > a_i) \geq \prod_{i=1}^m P(X_i > a_i)$$

for all $a_1, \ldots, a_m$, then in particular:

$$P(\bigcap_{i=1}^m X_i > a_i^*) \geq \prod_{i=1}^m S_{X_i}(a_i^*) = (1+\theta)^{-m}.$$

Hence, in this case a sufficient condition for Eq. (10) to hold is that:

$$(1+\theta)^{-m} \geq \alpha. \tag{13}$$

Note that the condition Eq. (13) is satisfied whenever the risk premium charged by the reinsurance company, i.e., θ, is not too large. If this condition is *not satisfied*, the cedent has little or nothing to gain, and should not buy a reinsurance contract.

Note also that independence is a special case of positively upper orthant dependence. Thus, the condition Eq. (13) covers the case where $X_1, \ldots, X_m$ are independent as well.

[c] Strictly speaking the result from Huseby and Christensen (2020) was formulated for the case with independent risks, in which case the condition Eq. (10) can be expressed more explicitly as Eq. (13). However, in the proof only the general condition given here was applied.

Obviously, the condition Eq. (11) uniquely determines the optimal values for $a_1, \ldots, a_m$. Furthermore, by Theorem 2.1 the optimization problem can be reformulated as an optimization problem with respect to the remaining unknown contract parameters $b_1, \ldots, b_m$ subject to a single constraint:

Theorem 2.2. *Assume that $a_1^* \ldots, a_m^*$, given by Eq. (11), satisfies Eq. (10). Then the remaining optimal contract parameters $b_1^*, \ldots, b_m^*$ can be found by solving the following optimization problem:*

$$\textit{Minimize:} \quad \sum_{i=1}^{m} E[R_i(X_i)] \tag{14}$$

$$\textit{Subject to:} \quad P(\boldsymbol{X} \in \mathcal{C}) = \alpha \tag{15}$$

with respect to $b_1, \ldots, b_m$.

Proof: We start out by noting that the constraint Eq. (15) is the same as the condition Eq. (12). Thus, by Theorem 2.1 this constraint is justified. Furthermore, by the definition of the set $\mathcal{C}$ given in Eq. (9), the constraint Eq. (15) can be written as:

$$P(\sum_{i=1}^{m} I_i(X_i) > \sum_{i=1}^{m} a_i^*) = \alpha$$

Then, by Eq. (3) it follows that under the constraint Eq. (15), the retained risk term is given by:

$$S^{-1}_{\sum_{i=1}^{m} I_i(X_i)}(\alpha) = \sum_{i=1}^{m} a_i^*.$$

Hence, the resulting total α-level value-at-risk becomes:

$$V_\alpha = \sum_{i=1}^{m} a_i^* + (1+\theta) \sum_{i=1}^{m} E[R_i(X_i)]$$

From this it follows that minimizing V_α is equivalent to minimizing $\sum_{i=1}^{m} E[R_i(X_i)]$ subject to the constraint Eq. (15) with respect to $b_1, \ldots, b_m$ ■

We observe that according to Eq. (11), the optimal values $a_1^*, \ldots, a_m^*$ satisfies:

$$S_{X_1}(a_1^*) = \cdots = S_{X_1}(a_m^*) = (1+\theta)^{-1}$$

We let A denote this common probability, i.e., $A = (1+\theta)^{-1}$. In the following it is also convenient to introduce new variables:

$$B_i = S_{X_i}(b_i) = P(X_i > b_i), \quad i = 1, \ldots, m,$$

and solve the optimization problem with respect to $B_1, \ldots, B_m$ instead of $b_1, \ldots, b_m$. Denoting the optimal values by $B_1^*, \ldots, B_m^*$ we can of course find the corresponding optimal values for $b_1, \ldots, b_m$ by:

$$b_i^* = S_{X_i}^{-1}(B_i^*), \quad i = 1, \ldots, m.$$

2.1. *Handling the constraint*

In this subsection we will explain how to determine the set of B_i-values satisfying the constraint Eq. (15). Before we do this, however, we note that:

$$P(\boldsymbol{X} \in \mathcal{C}) \leq P(\bigcup_{i=1}^{m} X_i > b_i) \leq \sum_{i=1}^{m} B_i$$

Since $B_1, \ldots, B_m$ typically are small numbers (e.g., 0.01), this upper bound is often quite good. This means that the constraint is approximately linear in $B_1, \ldots, B_m$, at least for a moderately sized m. In the next section this property will be illustrated by numerical examples.

An easy way to handling the constraint Eq. (15) is by using Monte Carlo simulation. Assuming that we have generated N samples $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N$ from the distribution of $\boldsymbol{X}$, we can estimate $p_{\mathcal{C}} = P(\boldsymbol{X} \in \mathcal{C})$ for given values of $B_1, \ldots, B_m$, by computing the resulting fraction of samples in the set $\mathcal{C}$. The values $B_1, \ldots, B_m$ can then be adjusted so that this fraction becomes equal to the desired value α. If N is large, we obtain a stable estimate of $p_{\mathcal{C}}$. A challenge with this approach, however, is that the event $\{\boldsymbol{X} \in \mathcal{C}\}$ has a small probability of occurring. Hence, most of the simulations will be waisted on parts of the sample space which are not affected by the B_is. Thus, using this approach N needs to be quite large in order to obtain stable results.

In order to improve the precision, we assume that we can find a set $\mathcal{D}$ such that $\mathcal{C} \subset \mathcal{D}$ for all relevant values of $B_1, \ldots, B_m$, and such that $p_{\mathcal{D}} = P(\boldsymbol{X} \in \mathcal{D})$ is known. Moreover, we let $p_{\mathcal{C}|\mathcal{D}} = P(\boldsymbol{X} \in \mathcal{C} | \boldsymbol{X} \in \mathcal{D})$. Since $\mathcal{C} \subset \mathcal{D}$, we get

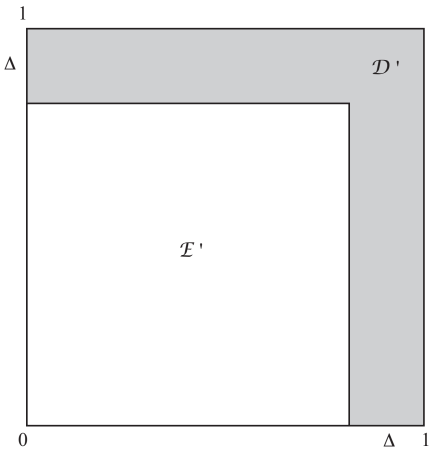

Fig. 2. $\mathcal{D}' = \{\boldsymbol{u} : 1 - \Delta < u_i < 1,\, i = 1, 2\}$

that:

$$p_{\mathcal{C}} = p_{\mathcal{C}|\mathcal{D}} \cdot p_{\mathcal{D}}.$$

We then generate N samples $\boldsymbol{X}_1, \dots \boldsymbol{X}_N$ from the conditional distribution of $\boldsymbol{X}$ given that $\boldsymbol{X} \in \mathcal{D}$, and estimate $p_{\mathcal{C}|\mathcal{D}}$ by:

$$\hat{p}_{\mathcal{C}|\mathcal{D}} = \frac{1}{N} \sum_{k=1}^{N} I(\boldsymbol{X}_k \in \mathcal{C})$$

The resulting estimate of the unconditional probability $p_{\mathcal{C}}$ is then given by:

$$\hat{p}_{\mathcal{C}} = \hat{p}_{\mathcal{C}|\mathcal{D}} \cdot p_{\mathcal{D}}.$$

We assume that $\boldsymbol{X}_1, \dots, \boldsymbol{X}_N$ are generated by transforming independent variables $\boldsymbol{U}_1, \dots, \boldsymbol{U}_N$:

$$\boldsymbol{X}_k = \psi(\boldsymbol{U}_k), \quad k = 1, \dots, N,$$

where ψ is strictly increasing in each argument, and where the variables $\boldsymbol{U}_1, \dots, \boldsymbol{U}_N$ are sampled uniformly from the set $\mathcal{D}'$ given by:

$$\mathcal{D}' = \{\boldsymbol{u} : 1 - \Delta < u_i < 1,\, i = 1, \dots, m\}.$$

Figure 2 shows $\mathcal{D}'$ for the case where $m = 2$. We may think of $\boldsymbol{U}_1, \dots, \boldsymbol{U}_N$ as variables distributed uniformly on $[0, 1]^m$, but sampled from the conditional distribution restricted to the set $\mathcal{D}'$.

Finally, we let:

$$\mathcal{D} = \{\boldsymbol{x} = \psi(\boldsymbol{u}) : \boldsymbol{u} \in \mathcal{D}'\}.$$

The quantity Δ is chosen as small as possible but still large enough so that $\mathcal{C} \subset \mathcal{D}$. The specific value of Δ depends on the joint distribution of $\boldsymbol{X}$, but in most cases $\Delta = 2\alpha$ works fine.

Sampling $\boldsymbol{U}_1, \dots, \boldsymbol{U}_N$ uniformly from $\mathcal{D}'$ is not difficult to accomplish, but we skip the details here. Furthermore, the transformation ψ can be constructed from the inverse distribution functions and inverse conditional distribution functions of $X_1, \dots, X_m$. It is straightforward to verify that this ensures that $\boldsymbol{X}_1, \dots, \boldsymbol{X}_N$ become distributed according to the conditional distribution of $\boldsymbol{X}$ given that $\boldsymbol{X} \in \mathcal{D}$ as claimed. As a result the stability of the probability estimates are greatly improved.

2.2. *The objective function*

We then consider the objective function given in Eq. (14), and let f_{X_i} denote the density of X_i, $i = 1, \dots, m$, which we denote by:

$$\Phi = \sum_{i=1}^{m} \Phi_i = \sum_{i=1}^{m} E[R_i(X_i)]$$

where we for $i = 1, \dots, m$ have:

$$\begin{aligned}\Phi_i = E[R_i(X_i)] &= \int_{a_i}^{b_i} (x - a_i) f_{X_i}(x) dx \\ &+ (b_i - a_i) P(X_i > b_i) \\ &= \int_{a_i}^{b_i} x f_{X_i}(x) dx \\ &+ b_i P(X_i > b_i) - a_i P(X_i > a_i).\end{aligned}$$

$\Phi_1, \dots, \Phi_m$ and hence also Φ can easily be calculated as functions of $B_1, \dots, B_m$ using numerical integration. Alternatively, it is possible to estimate these functions by using Monte Carlo simulation.

To get a better overview of the possible solutions to the optimization problem, we introduce the superlevel and sublevel sets of the objective function Φ expressed in terms of $\boldsymbol{B} = (B_1, \dots, B_m)$. That is, we let:

$$L_c^+(\Phi) = \{\boldsymbol{B} \in [0, 1]^m : \Phi(\boldsymbol{B}) \geq c\}$$

$$L_c^-(\Phi) = \{\boldsymbol{B} \in [0, 1]^m : \Phi(\boldsymbol{B}) \leq c\}$$

The sets $L_c^+(\Phi)$ and $L_c^-(\Phi)$ are referred to as respectively the superlevel and sublevel sets of the function Φ relative to the level c. A function is said to be *quasiconvex* if all its sublevel sets are convex, while a function is said to be *quasiconcave* if all its superlevel sets are convex. See Boyd and Vandenberghe (2004).

The following result provides sufficient conditions for quasiconvexity and quasiconcavity of Φ:

Proposition 2.1. *If $\Phi_1(B_1), \dots, \Phi_m(B_m)$ are convex functions, then Φ is a quasiconvex function of $\boldsymbol{B}$. If $\Phi_1(B_1), \dots, \Phi_m(B_m)$ are concave functions, then Φ is a quasiconcave function of $\boldsymbol{B}$.*

Proof: To prove the first claim, we assume that $\Phi_1(B_1), \dots, \Phi_m(B_m)$ are convex functions, and let $\boldsymbol{B}^{(j)} = (B_1^{(j)}, \dots, B_m^{(j)}) \in L_c^-(\Phi)$, $j = 1, 2$. In order to show that $L_c^-(\Phi)$ is convex, we must show that for any $\lambda \in [0, 1]$, we also have that $\boldsymbol{B} = \lambda \boldsymbol{B}^{(1)} + (1 - \lambda)\boldsymbol{B}^{(2)} \in L_c^-(\Phi)$.

Since $\Phi_1, \dots, \Phi_m$ are convex, we know that for $i = 1, \dots, m$ we have:

$$\begin{aligned}\Phi_i(\lambda B_i^{(1)} &+ (1-\lambda)B_i^{(2)}) \\ &\leq \lambda\Phi_i(B_i^{(1)}) + (1-\lambda)\Phi_i(B_i^{(2)})\end{aligned}$$

Hence, we get that:

$$\begin{aligned}\Phi(\boldsymbol{B}) &= \Phi(\lambda\boldsymbol{B}^{(1)} + (1-\lambda)\boldsymbol{B}^{(2)}) \\ &= \sum_{i=1}^{m} \Phi_i(\lambda B_i^{(1)} + (1-\lambda)B_i^{(2)}) \\ &\leq \sum_{i=1}^{m} \lambda\Phi_i(B_i^{(1)}) + (1-\lambda)\Phi_i(B_i^{(2)}) \\ &= \lambda\Phi(\boldsymbol{B}^{(1)}) + (1-\lambda)\Phi(\boldsymbol{B}^{(2)}) \\ &\leq \lambda c + (1-\lambda)c = c.\end{aligned}$$

Thus, we conclude that $\boldsymbol{B} \in L_c^-(\Phi)$, i.e., $L_c^-(\Phi)$ is convex. The second claim is proved in a completely similar way ■

Using the expressions for $\Phi_1, \dots, \Phi_m$ it is easy to see that we have:

$$\frac{\partial\Phi_i}{\partial b_i} = P(X_i > b_i), \quad i = 1, \dots, m. \tag{16}$$

We can also derive the partial derivatives with respect to $B_1, \dots, B_m$, which are given by:

$$\frac{\partial\Phi_i}{\partial B_i} = -\frac{B_i}{f_{X_i}(S_{X_i}^{-1}(B_i))}, \quad i = 1, \dots, m. \tag{17}$$

A function is convex if its partial derivative is increasing. Thus, Φ_i is convex if:

$$-\frac{B_i}{f_{X_i}(S_{X_i}^{-1}(B_i))} \quad \text{is increasing in } B_i.$$

or equivalently if:

$$\frac{f_{X_i}(S_{X_i}^{-1}(B_i))}{B_i} \quad \text{is increasing in } B_i.$$

We then substitute $B_i = S_{X_i}(x)$. Since B_i is a decreasing function of x, it follows that Φ_i is convex if:

$$\frac{f_{X_i}(x)}{S_{X_i}(x)} \quad \text{is decreasing in } x.$$

Similarly, it follows that Φ_i is concave if:

$$\frac{f_{X_i}(x)}{S_{X_i}(x)} \quad \text{is increasing in } x.$$

We recognize the ratio $f_{X_i}(x)/S_{X_i}(x)$ as the *hazard rate* of the distribution of X_i. The following theorem summarizes these findings:

Theorem 2.3. *If the risks $X_1, \dots, X_m$ have decreasing hazard rates, then Φ is a quasiconvex function of $\boldsymbol{B}$. If $X_1, \dots, X_m$ have increasing hazard rates, then Φ is a quasiconcave function of $\boldsymbol{B}$.*

3. Numerical examples

In this section, we will illustrate the results from the previous section by presenting a few examples. Only bivariate cases will be considered here, i.e., $m = 2$, and we let $\alpha = 0.01$ and $\theta = 0.2$. The examples are illustrated with plots showing iso-curves for the objective function Φ, and constraint curves. The iso-curves are calculated using numerical integration, while the constraint curves are calculated using the importance sampling method explained in Subsection 2.1 with $N = 1000000$ simulations.

In the first example, we let X_1 and X_2 be independent and Pareto distributed with mean 50 and standard deviation 70. The Pareto distribution has a decreasing hazard rate. Thus, it follows from Theorem 2.3 that Φ is quasiconvex, and hence, that the sublevel sets are convex. Figure 3 shows iso-curves for the objective function Φ along with the constraint curve. The iso-curves clearly show that indeed the sublevel sets are convex. The constraint curve is bending slightly away from the origin. The optimal combination of B_1 and B_2 is the point of the constraint curve where the objective function is smallest. From the figure it is evident that this corresponds to a *balanced solution*, i.e. a solution where $B_1 = B_2$. Detailed calculations yield the solution $(B_1^*, B_2^*) = (0.0051, 0.0051)$. The corresponding solution for the b_is is $(b_1^*, b_2^*) = (294, 294)$.

As long as the constraint curve is either approximately linear or slightly bending away from the origin, a balanced solution will always be optimal when X_1 and X_2 have the same distribution and this distribution has a *decreasing hazard rate*. This even includes the case where X_1 and X_2 are exponentially distributed, even though the isocurves are linear in this case. The reason for this, is that the constraint curve is bending slightly away from the origin in the exponential case as well, and thus, touches the isocurve for the minimal value at a unique point, which by symmetry will correspond to a balanced solution.

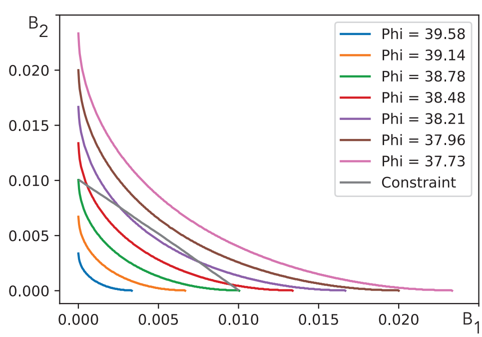

Fig. 3. Constraint and Iso-contours where X_1 and X_2 are Pareto distributed with mean 50 and standard deviation 70.

In the second example, we let X_1 and X_2 be independent and truncated normally distributed with mean 50 and standard deviation 30. The truncated normal distribution has an increasing hazard rate. Thus, by Theorem 2.3 Φ is quasiconcave, and hence, that the superlevel sets are convex. Figure 4 shows iso-curves for the objective function Φ along with the constraint curve. We observe that the superlevel sets are indeed convex, while the constraint curve is approximately linear. In this case, optimal solutions will be found at the boundary of the set of possible values. By symmetry, there will be two optimal solutions, one where $B_1 = 0$ and B_2 is between 0.01 and 0.0125, and another where $B_2 = 0$ and B_1 is between 0.01 and 0.0125. Detailed calculations yield the two solutions $(B_1^*, B_2^*) = (0, 0.011)$ and $(B_1^*, B_2^*) = (0.011, 0)$. The corresponding solutions for the b_is are $(b_1^*, b_2^*) = (\infty, 127)$ and $(b_1^*, b_2^*) = (127, \infty)$.

As long as the constraint curve is either approximately linear or slightly bending against the origin, a strongly unbalanced solution like the one above will always be optimal when X_1 and X_2 have the same distribution and this distribution has an *increasing hazard rate*. Such cases include e.g., Gamma distributions with shape parameters greater than one, as well as Weibull-distributions with shape parameters greater than one.

Note that in such cases Lagrange optimization will produce a solution where the objective function is *maximized*. Thus, by using this method, we will end up with the worst possible solution.

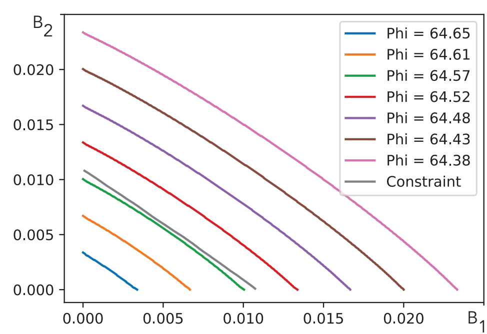

Fig. 4. Constraint and Iso-contours where X_1 and X_2 are truncated normally distributed with mean 50 and standard deviation 30.

In the final example, we let X_1 and X_2 be independent and Pareto distributed, both with mean 50. In this case, however, the standard deviation of X_1 is 60, while the standard deviation of X_2 is 40. The objective function Φ is still quasiconvex, and the sublevel sets are convex. Figure 5 shows iso-curves for the objective function Φ along with the constraint curve. The constraint curve is once again bending slightly away from the origin. Since the two risks have different distributions, the optimal combination of B_1 and B_2 is not balanced. Still, it corresponds to a unique point where the constraint curve touches an iso-curve. The exact location of the point can be found either by Lagrange optimization, or by a simple search along the constraint curve. Detailed calculations yield the solution $(B_1^*, B_2^*) = (0.0066, 0.0036)$. The corresponding solution for the b_is is $(b_1^*, b_2^*) = (250, 268)$.

Note that since B_1^* is almost twice as large as B_2^*, the risk X_2 gets a much better reinsurance coverage than the risk X_1. The reason for this is that since the standard deviation of X_2 is less than the standard deviation for X_1, it is cheaper to reinsure X_2 than X_1.

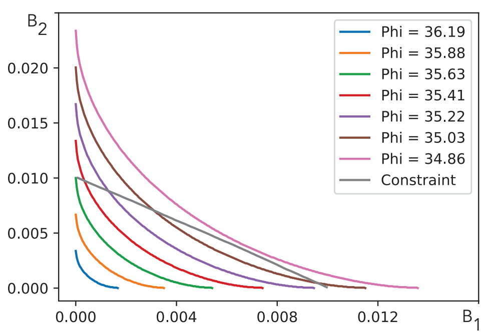

Fig. 5. Constraint and Iso-contours where X_1 is Pareto distributed with mean 50 and standard deviation 60, while X_2 is Pareto distributed with mean 50 and standard deviation 40.

4. Conclusions and further work

In the present paper, we have seen how multiple reinsurance contracts can be optimized with respect to value-at-risk. In particular, we have proved that the optimal solution depends on monotonicity properties of the hazard rates of the risk distributions. We have also presented an efficient simulation method based on importance sampling. The proposed methodology is illustrated by some numerical examples.

Future work in this area includes optimization with respect to other risk measures and objective functions, how to handle dependent risks, as well as a study of how the results may change when the risk distribution parameters are subject to uncertainty.

Acknowledgement

The author is grateful to Ingrid Hobæk Haff for introducing the problem of multivariate reinsurance. We also wish to thank Yinzhi Wang and Xiaofei Li who have worked together with me on several related projects.

References

Boyd, S. and L. Vandenberghe (2004). *Convex Optimization*. Cambridge University Press.

Bühlmann, H. (1980). An economic premium principle. *ASTIN Bulletin: The Journ. of IAA* (11), 52–60.

Cheung, K., K. Sung, S. Yam, and S. P. Yung (2014). Optimal reinsurance under general law-invariant risk measures. *Scand. Actuarial Journal* (1), 72–91.

Chi, Y. C., X. S. Lin, and K. S. Tan (2017). Optimal reinsurance under the risk-adjusted value of an insurer's liability and an economic reinsurance premium principle. *North American Actuarial Journal* (21 (3)), 417–432.

Cong, J. F. and K. S. Tan (2016). Optimal var-based risk management with reinsurance. *Annals of Operations Research* (237), 177–202.

Furman, E. and R. Zitikis (2007). An actuarial premium pricing model for nonnormal insurance and financial risks in incomplete markets. *North American Actuarial Journal* (11(3)), 174–176.

Huseby, A. B. and D. Christensen (2020). Optimal reinsurance contracts in the multivariate case. In P. Baraldi, F. P. D. Maio, and E. Zio (Eds.), *e-proceedings of the 30th European Safety and Reliability Conference and 15th Probabilistic Safety Assessment and Management Conference (ESREL2020 PSAM15)*, pp. 1–8. Research Publishing Services.

Lu, Z., L. Liu, and S. Meng (2013). Optimal reinsurance with concave ceded loss functions under VaR and CTE risk measures. *Insurance: Mathematics and Economics* (52), 46–51.

Shaked, M. (1982). A general theory of some positive dependence notions. *Journal of Multivariate Analysis* (12), 199–218.