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Optimal Reinsurance Contracts under Conditional Value-at-risk

Kristina Rognlien Dahl

Department of Mathematics, University of Oslo, Norway. E-mail: kristrd@math.uio.no

Arne Bang Huseby

Department of Mathematics, University of Oslo, Norway. E-mail: arne@math.uio.no

Marius Helvig Havgar

Department of Mathematics, University of Oslo, Norway. E-mail: Marius.Havgar@wilhelmsen.com

An insurance contract implies that risk is ceded from ordinary policy holders to companies. However, companies do the same thing between themselves, and this is known as reinsurance. The problem of determining reinsurance contracts which are optimal with respect to some reasonable criterion has been studied extensively within actuarial science. Different contract types are considered such as stop-loss contracts where the reinsurance company covers risk above a certain level, and insurance layer contracts where the reinsurance company covers risk within an interval. The contracts are then optimized with respect to some risk measure, such as value-at-risk or conditional value-at-risk. In the present paper we consider the problem of minimizing conditional value-at-risk in the case of multiple stop-loss contracts. Such contracts are known to be optimal in the univariate case, and the optimal contract is easily determined. We show that the same holds in the multivariate case, both with dependent and independent risks. The results are illustrated with some numerical examples.

Keywords: optimal reinsurance, multivariate optimization, conditional value-at-risk, Monte Carlo methods

1. Introduction

In general, insurance is purchased in order to reduce the risk of large losses, where the insurer will cover certain losses. The party that hedges against such losses will normally pay a risk premium for this protection, typically a fraction of the expected value of the insurer's cost. The same practice is common among insurance companies. When an insurance company buys insurance for a part of an insurance contract from another company, this is known as *reinsurance*. We refer to the party which hedges against losses by purchasing reinsurance as the *cedent*. The cedent purchases reinsurance from the *reinsurer*.

The problem of determining reinsurance contracts which are optimal with respect to some reasonable criterion has been studied extensively within actuarial science. Different contact types are considered such as *stop-loss contracts* and *insurance layer contracts*. The contracts can then be optimized with respect to some risk measure such as *value-at-risk* or *conditional value-at-risk*. In the univariate case, i.e., when only a single risk is reinsured, solutions to several variations of the optimization problem can be found in Cheung et al. (2014). In particular, it is shown that if value-at-risk is used as risk measure, the optimal contract is an insurance layer contract, while if conditional value-at-risk is used, the optimal contract is a stop-loss contract. Some other recent work in this area are Lu et al. (2013), Cong and Tan (2016), and Chi et al. (2017).

The topic of the present paper is the case where the cedent has multiple risks which cannot be bundled together. Instead the risks must be covered by separate reinsurance contracts. This problem was considered in Huseby and Christensen (2020) for the case where value-at-risk was used as risk measure. It was shown that optimal solutions must satisfy certain conditions. These conditions simplify the optimization problem significantly. Huseby (2022) applied these conditions and proved that the optimization problem can be expressed as an optimization problem with just n variables subject to a single constraint.

In the present paper, we consider the case where conditional value-at-risk is used as risk measure. This risk measure can be viewed as a limiting case of a generalized version of this measure. We show that except for the limiting case, the generalized conditional value-at-risk measure behaves similar to the value-at-risk measure. In particular, optimal insurance layer contracts can be found using the same methods as presented in Huseby and Christensen (2020) and Huseby (2022). The optimal solution for the conditional value-at-risk measure can then be obtained as a limit of insurance layer contracts. In particular, we show that this limit can be chosen to be a stop-loss contract. The methods are illustrated by numerical examples.

2. Risk measures

In this section we introduce the risk measures we will use in this paper. The risk measures will typically be defined relative to some random variable X. In order to avoid technical issues we only consider *non-negative* risks with *finite expectations*. The cumulative distribution function of X is denoted by $F_X(x) = P(X \leq x)$. We start out by reviewing some basic properties of *value-at-risk*.

The α-level value-at-risk associated with the risk X, denoted $VaR_\alpha(X)$, is defined for $0 \leq \alpha \leq 1$ as the α-percentile of the distribution of X. Thus, $VaR_\alpha(X) = F_X^{-1}(\alpha)$, where:

$$F_X^{-1}(\alpha) = \inf\{x \ : \ P(X \leq x) \geq \alpha\}. \quad (1)$$

If F_X is strictly increasing, it is easy to show that:

$$VaR_\alpha(X) = r \text{ if and only if } P(X < r) \leq \alpha \leq P(X \leq r). \quad (2)$$

In particular, for F_X strictly increasing, the following holds true:

$$\text{If } P(X \leq r) = \alpha, \text{ then } F_X^{-1}(\alpha) = r. \quad (3)$$

It is well-known that the value-at-risk function has the property that if $a > 0$, then:

$$VaR_\alpha(a \cdot X + b) = a \cdot VaR_\alpha(X) + b \quad (4)$$

We observe that $VaR_\alpha(X)$ does not include information about the right hand tail of the distribution of X beyond the α-percentile. From a risk management point of view the tail area is often of interest. A risk measure which includes more tail information is the α, β-level *conditional value-at-risk*, denoted $CVaR_{\alpha,\beta}$, and defined for $0 \leq \alpha < \beta \leq 1$ by:

$$\begin{aligned} CVaR_{\alpha,\beta}(X) &= \frac{1}{\beta - \alpha} \int_\alpha^\beta VaR_p(X)\, dp \\ &= \frac{1}{\beta - \alpha} \int_\alpha^\beta F_X^{-1}(p)\, dp \quad (5) \end{aligned}$$

By Eq. (4) it follows that we also have:

$$CVaR_{\alpha,\beta}(aX + b) = aCVaR_{\alpha,\beta}(X) + b \quad (6)$$

for all $a > 0$. Moreover, by substituting $p = F_X(x)$, we obtain the following alternative expression for $CVaR_{\alpha,\beta}(X)$:

$$CVaR_{\alpha,\beta}(X) = \frac{1}{\beta - \alpha} \int_{F_X^{-1}(\alpha)}^{F_X^{-1}(\beta)} x\, dF_X(x) \quad (7)$$

A special case of the α, β-level conditional value-at-risk measure is obtained by letting $\beta = 1$. We refer to this risk measure as the α-level *conditional value-at-risk*, denoted by $CVaR_\alpha(X)$. Thus, we have:

$$\begin{aligned} CVaR_\alpha(X) &= \frac{1}{1 - \alpha} \int_\alpha^1 VaR_p(X)\, dp \\ &= \frac{1}{1 - \alpha} \int_\alpha^1 F_X^{-1}(p)\, dp \quad (8) \end{aligned}$$

Since we have assumed that $E(X) < \infty$, this risk measure is finite, and as above, we obtain an alternative expression for $CVaR_\alpha(X)$ by substituting $p = F_X(x)$:

$$CVaR_\alpha(X) = \frac{1}{1 - \alpha} \int_{F_X^{-1}(\alpha)}^{\infty} x\, dF_X(x) \quad (9)$$

3. Multivariate reinsurance contracts

We now consider the problem of optimizing insurance contracts in the multivariate case. Thus, we let $X_1, \ldots, X_m$ be m non-negative random variables representing the risks from m business lines[a]. The cumulative distribution function of X_i, is denoted by F_{X_i}, $i = 1, \ldots, m$. As before we

[a] Note that the risks $X_1, \ldots, X_m$ are *not* assumed to be independent at this stage.

assume that $E(X_i) < \infty$, $i = 1, \ldots, m$. To avoid technical issues we also assume that $X_1, \ldots, X_m$ are absolutely continuously distributed, and that F_{X_i} is strictly increasing, $i = 1, \ldots, m$.

All risks are reinsured separately. For the ith risk, $R_i(X_i)$ denotes the cost covered by the reinsurance company, and $I_i(X_i) = X_i - R_i(X_i)$ denotes the cost covered by the cedent, $i = 1, \ldots, m$.

The price paid by the cedent for the ith contract is denoted by π_{X_i}, and is assumed to be of the form:

$$\pi_i = (1+\theta)E[R_i(X_i)], \quad i = 1, \ldots, m.$$

where $\theta > 0$ represents the risk premiums charged by the reinsurance company for handling the risks. The total cost covered by the cedent is then:

$$\Psi = \sum_{i=1}^{m} I_i(X_i) + (1+\theta)\sum_{i=1}^{m} E[R_i(X_i)].$$

By Eq. (6) the resulting α, β-level conditional value-at-risk of Ψ, is given by:

$$CVaR_{\alpha,\beta}(\Psi) = CVaR_{\alpha,\beta}(\sum_{i=1}^{m} I_i(X_i)) + (1+\theta)\sum_{i=1}^{m} E[R_i(X_i)].$$

We observe that $CVaR_{\alpha,\beta}(\Psi)$ consists of two terms. We refer to the first term,

$$CVaR_{\alpha,\beta}(\sum_{i=1}^{m} I_i(X_i))$$

as the *retained risk term*, while:

$$\sum_{i=1}^{m} \pi_i = (1+\theta)\sum_{i=1}^{m} E[R_i(X_i)]$$

is referred to as the *premium term*.

If the risks are reinsured with *insurance layer* contracts, the functions $R_1 \ldots, R_m$ are given by:

$$R_i(X_i) = \begin{cases} 0 & \text{for } X_i < a_i \\ X_i - a_i & \text{for } a_i \leq X_i \leq b_i \\ b_i - a_i & \text{for } X_i > b_i \end{cases} \quad (10)$$

where $a_i < b_i$. Moreover, the functions $I_1 \ldots, I_m$ are given by:

$$I_i(X_i) = \begin{cases} X_i & \text{for } X_i < a_i \\ a_i & \text{for } a_i \leq X_i \leq b_i \\ X_i - (b_i - a_i) & \text{for } X_i > b_i \end{cases} \quad (11)$$

The main objective is to find $a_1, b_1, \ldots, a_m, b_m$ so that $CVaR_{\alpha,\beta}(\Psi)$ is minimized.

In limiting cases where $b_i = \infty$, $i = 1, \ldots, m$, the above functions can be simplified to:

$$R_i(X_i) = \max(X_i - a_i, 0)$$

$$I_i(X_i) = \min(X_i, a_i)$$

Such contracts are referred to as *stop-loss* contracts.

4. Optimizing reinsurance contracts

Huseby and Christensen (2020) considered the problem of optimizing multivariate reinsurance contracts using the α-level value-at-risk, and proved the following key result:

Theorem 4.1. *Assume that* $a_1^*, b_1^*, \ldots, a_m^*, b_m^*$ *are optimal contract parameter values for the* α*-level value-at-risk measure, and that:*

$$P(\bigcap_{i=1}^{m} X_i > a_i^*) \geq 1 - \alpha. \quad (12)$$

Then the following conditions must hold true:

$$a_i^* = F_{X_i}^{-1}(\frac{\theta}{1+\theta}), \quad i = 1, \ldots, m, \quad (13)$$

and:

$$P(\sum_{i=1}^{m} I_i(x_i) \leq \sum_{i=1}^{m} a_i^*) = \alpha. \quad (14)$$

We now prove the corresponding result for the α, β-level conditional value-at-risk:

Theorem 4.2. *Assume that* $a_1^*, b_1^*, \ldots, a_m^*, b_m^*$ *are optimal contract parameter values for the* α, β*-level conditional value-at-risk measure, and that* $a_1^*, \ldots, a_m^*$ *satisfy Eq. (12). Then the following conditions must hold true:*

$$a_i^* = F_{X_i}^{-1}(\frac{\theta}{1+\theta}), \quad i = 1, \ldots, m, \quad (15)$$

and:

$$P(\sum_{i=1}^{m} I_i(X_i) \leq \sum_{i=1}^{m} a_i^*) = \beta. \tag{16}$$

Proof: The proof of this result is very similar to the proof of Theorem 4.1, so we just include the main arguments here. For further details we refer to Huseby and Christensen (2020).

We start out by assuming that the values of $a_1, \ldots, a_m$ are chosen so that:

$$P(\bigcap_{i=1}^{m} X_i > a_i) \geq 1 - \alpha. \tag{17}$$

Since $X_1, \ldots, X_m$ are assumed to be absolutely continuously distributed, it is easy to see that for the given values $a_1, \ldots, a_m$ there must exist values $b_i > a_i$, $i = 1, \ldots, m$ such that:

$$P(\sum_{i=1}^{m} I_i(X_i) \leq \sum_{i=1}^{m} a_i) = \beta. \tag{18}$$

We then assume that $b_1, \ldots, b_m$ are chosen according to this condition (i.e., dependent on the given values $a_1, \ldots, a_m$). By Eq. (3) this implies that:

$$VaR_\beta(\sum_{i=1}^{m} I_i(X_i)) = \sum_{i=1}^{m} a_i.$$

If $p \in [\alpha, \beta]$, the condition Eq. (18) also implies that:

$$P(\sum_{i=1}^{m} I_i(X_i) \leq \sum_{i=1}^{m} a_i) \geq p.$$

At the same time, it follows by Eq. (17) that:

$$\begin{aligned} P(\sum_{i=1}^{m} I_i(X_i) < \sum_{i=1}^{m} a_i) &\leq P(\bigcup_{i=1}^{m} X_i \leq a_i) \\ &= 1 - P(\bigcap_{i=1}^{m} X_i > a_i) \leq 1 - (1-\alpha) \leq p. \end{aligned}$$

By Eq. (2) this implies that:

$$VaR_p(\sum_{i=1}^{m} I_i(X_i)) = \sum_{i=1}^{m} a_i, \text{ for all } p \in [\alpha, \beta].$$

Hence, it follows that:

$$\begin{aligned} &CVaR_{\alpha,\beta}(\sum_{i=1}^{m} I_i(X_i)) \\ &= \frac{1}{\beta - \alpha} \int_\alpha^\beta VaR_p(\sum_{i=1}^{m} I_i(X_i))\, dp \\ &= \sum_{i=1}^{m} a_i. \end{aligned} \tag{19}$$

If we increase the b_is, it is easy to show, using similar arguments as above, that we still have:

$$CVaR_{\alpha,\beta}(\sum_{i=1}^{m} I_i(X_i)) = \sum_{i=1}^{m} a_i.$$

At the same time premium term increases. As a result $CVaR_{\alpha,\beta}(\Psi)$ is increased.

If we decrease the values of the b_is, the retained risk term increases. At the same time the premium decreases. It can be shown, however, that the reduction in the premium term does not cover the increase in the retained risk term. Hence, we conclude that for the given values $a_1, \ldots, a_m$, the corresponding values $b_1, \ldots, b_m$ should be chosen so that Eq. (18) holds.

We now turn to determining the optimal values for $a_1, \ldots, a_m$, assuming that $b_1, \ldots, b_m$ are chosen accordingly so that Eq. (18) holds. By Eq. (19) it follows that we have:

$$CVaR_{\alpha,\beta}(\Psi) = \sum_{i=1}^{m} a_i + \sum_{i=1}^{m} \pi_i$$

where:

$$\begin{aligned} \pi_i = (1+\theta) &\int_{a_i}^{b_i} (x - a_i) dF_{X_i}(x) \\ &+ (b_i - a_i) P(X_i > b_i), \quad i = 1, \ldots, m. \end{aligned}$$

Using this, it is easy to see that for $i = 1, \ldots, m$ we have:

$$\frac{\partial}{\partial a_i} CVaR_{\alpha,\beta}(\Psi) = 1 - (1+\theta) P(X_i > a_i).$$

The optimal values for $a_1, \ldots, a_m$ can then be found by solving the equations:

$$\frac{\partial}{\partial a_i} CVaR_{\alpha,\beta}(\Psi) = 0, , \quad i = 1, \ldots, m,$$

yielding the solution Eq. (15) ■

Note that the optimal values for $a_1^*, \ldots, a_m^*$ for the α, β-level conditional value-at-risk measure are exactly the same as for the α-level value-at-risk measure. Moreover, the right-hand side of Eq. (16) is β, while the corresponding value in Eq. (14) is α. It is easy to see that this implies that an optimal solution for the β-level value-at-risk will also be optimal for the α, β-level conditional value-at-risk. Thus, the optimization results for the value-at-risk measure given in Huseby (2022) applies without any modifications to the α, β-level conditional value-at-risk as well. That is, we have the following result:

Theorem 4.3. *Assume that $a_1^* \ldots, a_m^*$, given by Eq. (15), satisfies Eq. (12). Then the remaining optimal contract parameters $b_1^*, \ldots, b_m^*$ for the α, β-level conditional value-at-risk can be found by solving the following optimization problem:*

$$\textit{Minimize:} \quad \sum_{i=1}^{m} E[R_i(X_i)] \tag{20}$$

$$\textit{Subject to:} \quad P(\sum_{i=1}^{m} I_i(X_i) \leq \sum_{i=1}^{m} a_i^*) = \beta \tag{21}$$

with respect to $b_1, \ldots, b_m$.

Proof: The proof of this result is identitical to the proof of the corresponding result for value-at-risk given in Huseby (2022). ■

We will describe the optimization procedure in further detail in the next section and illustrate this by numerical examples.

By using Theorem 4.2, it is easy to prove the corresponding result for α-level conditional value-at-risk. Since this risk measure is just a special case of the α, β-level conditional value-at-risk, obtained by letting $\beta = 1$, we see that for the α-level conditional value-at-risk, the condition Eq. (16) is simplified to:

$$P(\sum_{i=1}^{m} I_i(X_i) \leq \sum_{i=1}^{m} a_i^*) = 1. \tag{22}$$

This condition is obviously satisfied if we let $b_i = \infty$, $i = 1, \ldots, m$. The following theorem, which is an immediate consequence of Theorem 4.2, summarizes this:

Theorem 4.4. *Assume that $a_1^*, b_1^*, \ldots, a_m^*, b_m^*$ are optimal contract parameter values for the α-level conditional value-at-risk measure, and that $a_1^*, \ldots, a_m^*$ satisfy Eq. (12). Then optimal parameters can chosen as:*

$$a_i^* = F_{X_i}^{-1}(\frac{\theta}{1+\theta}), \quad i = 1, \ldots, m, \tag{23}$$

and:

$$b_i^* = \infty, \quad i = 1, \ldots, m. \tag{24}$$

That is, the optimal contracts can be chosen as stop-loss contracts ■

Note that if $P(X_i \leq M) = 1$, where $M < \infty$, then b_i^* may be chosen arbitrarily within the interval $[M, \infty)$. Thus, in such cases the optimal solution is not unique. For simplicity, however, we let $b_i^* = \infty$ since this solution is guaranteed to work anyway.

It is also worth noting that Theorem 4.4 actually provides a *fully specified optimal solution* for the α-level conditional value-at-risk. Thus, for this risk measure finding optimal contracts is actually much easier than for the α, β-level conditional value-at-risk, since there is no need to optimize the b_i^*-values further.

We observe that the condition Eq. (12) is needed for all the above results. If more information about the joint distribution of $X_1, \ldots, X_m$ is available, it is possible to obtain more explicit versions of this condition. This is shown in the following two propositions:

Proposition 4.1. *Assume that $X_1, \ldots, X_m$ are* positively upper orthant dependent. *Then a sufficient condition for Eq. (12) to hold is that:*

$$(1+\theta)^{-m} \geq 1 - \alpha. \tag{25}$$

Proof: According to Shaked (1982), $X_1, \ldots, X_m$ are said to be positively upper orthant dependent if:

$$P(\bigcap_{i=1}^{m} X_i > a_i) \geq \prod_{i=1}^{m} P(X_i > a_i)$$

for all $a_1, \dots, a_m$. Thus, it follows that

$$P(\bigcap_{i=1}^{m} X_i > a_i^*) \geq \prod_{i=1}^{m} P(X_i > a_i^*)$$
$$= (1+\theta)^{-m}.$$

Hence, a sufficient condition for Eq. (12) to hold is Eq. (25) as claimed ■

Note that independence is a special case of positively upper orthant dependence. Thus, the condition Eq. (25) covers the case where $X_1, \dots, X_m$ are independent as well. The second proposition covers the opposite situation, i.e., where the risks are strongly positively dependent.

Proposition 4.2. *Assume that* $X_1, \dots, X_m$ *are* comonotonic *risks. Then a sufficient condition for Eq. (12) to hold is that:*

$$(1+\theta)^{-1} \geq 1 - \alpha \tag{26}$$

Proof: $X_1, \dots, X_m$ are said to be comonotonic, if there exists a random variable Z and non-decreasing functions $h_1, \dots, h_m$ such that $(X_1, \dots, X_m) \stackrel{d}{=} (h_1(Z), \dots, h_m(Z))$. We denote the domain of Z by $\mathcal{Z}$, and introduce:

$$S_i = \{z \in \mathcal{Z} \mid h_i(z) \geq a_i^*\}, \quad i = 1, \dots, m.$$

Since $h_1, \dots, h_m$ are non-decreasing, there exists constants $c_1^*, \dots, c_m^*$ such that we either have $S_i = [c_i^*, \infty) \cap \mathcal{Z}$ or $S_i = (c_i^*, \infty) \cap \mathcal{Z}$, $i = 1, \dots, m$. From this it follows that there must exist some $k \in \{1, \dots, m\}$ such that:

$$S_k \subseteq S_i, \quad i = 1, \dots, m.$$

This implies that:

$$P(\bigcap_{i=1}^{m} X_i \geq a_i^*) = P(\bigcap_{i=1}^{m} Z \in S_i)$$
$$= P(Z \in S_k) = P(X_k \geq a_k^*)$$
$$= (1+\theta)^{-1} \geq 1 - \alpha$$

Hence, a sufficient condition for Eq. (12) to hold is Eq. (26) as claimed ■

If $X_1, \dots, X_m$ are comonotonic, it follows by well-known results that $X_1, \dots, X_m$ are associated random variables. Hence, it follows that $X_1, \dots, X_m$ are positively upper orthant dependent as well (see Shaked (1982)). Hence, Proposition 4.1 covers the comonotonic case as well. However, the condition Eq. (26) is less strict than Eq. (25). Thus, it makes sense to include the latter result as well.

We observe that the conditions Eq. (25) and Eq. (26) are satisfied whenever the risk premium, θ, charged by the reinsurance company, is not too large. If this condition is *not satisfied*, the cedent typically has little or nothing to gain, and should not reinsure the risks.

5. Numerical examples

In this section we will illustrate the results from the previous section by two numerical examples. All calculations are done by a combination of numerical integration and importance sampling. Before we present some numerical examples, we briefly review some of the results from Huseby (2022) addressing the optimization problem described in Theorem 4.3. We start out by focussing on the objective function which we denote by:

$$\Phi = \sum_{i=1}^{m} E[R_i(X_i)]$$

For the optimization it is convenient to express the objective function Φ in the terms of $\boldsymbol{B} = (B_1, \dots, B_m)$, where:

$$B_i = F_{X_i}(b_i) = P(X_i \leq b_i), \quad i = 1, \dots, m.$$

We also consider the *superlevel* and *sublevel* sets of the objective function defined respectively as:

$$L_c^+(\Phi) = \{\boldsymbol{B} \in [0,1]^m : \Phi(\boldsymbol{B}) \geq c\}$$
$$L_c^-(\Phi) = \{\boldsymbol{B} \in [0,1]^m : \Phi(\boldsymbol{B}) \leq c\}$$

A function is *quasiconvex* if all its sublevel sets are convex, while a function is *quasiconcave* if all its superlevel sets are convex. See Boyd and Vandenberghe (2004). If the objective function is quasiconvex, the solution is usually an inner point in the the set of feasible solutions, while if the objective function is quasiconcave, the solution will typically be located at the boundary of this set. In Huseby (2022) the following result was proved:

Theorem 5.1. *If the risks $X_1, \ldots, X_m$ have decreasing hazard rates, then Φ is a quasiconvex function of $\boldsymbol{B}$. If $X_1, \ldots, X_m$ have increasing hazard rates, then Φ is a quasiconcave function of $\boldsymbol{B}$.*

In the first example, X_1 and X_2 are independent and Pareto distributed with mean 50 and standard deviation 75. The Pareto distribution has a decreasing hazard rate. Thus, by Theorem 5.1 we have that Φ is quasiconvex, and hence, that the sublevel sets are convex. Figure 1 shows iso-curves for the objective function Φ along with the constraint curve for the α, β-level conditional value-at-risk, where $\alpha = 0.988$ and $\beta = 0.990$. The iso-curves clearly show that indeed the sublevel sets are convex. The constraint curve is bending slightly against the origin. The optimal combination of B_1 and B_2 is the point of the constraint curve where the objective function is smallest.

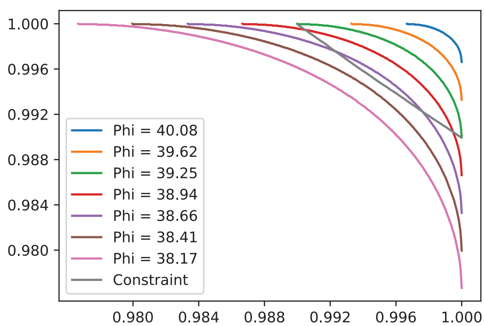

Fig. 1. Constraint and Iso-contours where X_1 and X_2 are Pareto distributed with mean 50 and standard deviation 75.

From the figure it is evident that this corresponds to a *balanced solution*, i.e. a solution where $B_1 = B_2$. Detailed calculations yield the solution $(B_1^*, B_2^*) = (0.995, 0.995)$. The corresponding solution for the b_is is $(b_1^*, b_2^*) = (303, 303)$.

If β is increased, the constraint curve will move further and further away from the origin and eventually end up as a single point $(B_1, B_2) = (1, 1)$. In Figure 2 we have shown how the α, β-level

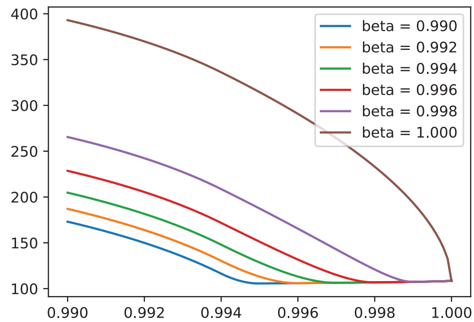

Fig. 2. Conditional value-at-risk for different β-values as a function of $B_1 = B_2$ when X_1 and X_2 are Pareto distributed with mean 50 and standard deviation 75.

conditional value-at-risk varies as a function of the common $B_1 = B_2$ value for different β-values, ranging from 0.990 to 1. We see that the optimal common value for B_1 and B_2 becomes larger and larger as β increases, and when $\beta = 1$, the optimal value for B_1 and B_2 is 1, as expected.

In the second example, X_1 and X_2 are independent and truncated normally distributed with mean 50 and standard deviation 25. The truncated normal distribution has an increasing hazard rate. Thus, by Theorem 5.1, Φ is quasiconcave, and hence, that the superlevel sets are convex. Figure 3 shows iso-curves for the objective function Φ along with the constraint curve for the α, β-level conditional value-at-risk, where $\alpha = 0.988$ and $\beta = 0.990$. The iso-curves clearly show that indeed the superlevel sets are convex. The constraint curve is almost linear. The optimal combination of B_1 and B_2 is the point of the constraint curve where the objective function is smallest.

From the figure it is evident that this corresponds to a *boundary solution*. Detailed calculations yield two solutions $(B_1^*, B_2^*) = (0.989, 1)$ and $(B_1^*, B_2^*) = (1, 0.989)$. The corresponding solutions for the b_is are respectively $(b_1^*, b_2^*) = (111, \infty)$ and $(b_1^*, b_2^*) = (\infty, 111)$.

If β is increased, the constraint curve will again move further and further away from the origin and eventually end up as a single point $(B_1, B_2) = (1, 1)$. In Figure 4 we have shown how the α, β-level conditional value-at-risk varies as a function

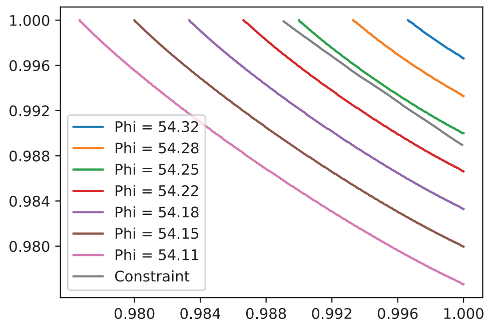

Fig. 3. Constraint and Iso-contours where X_1 and X_2 are truncated normally distributed with mean 50 and standard deviation 25.

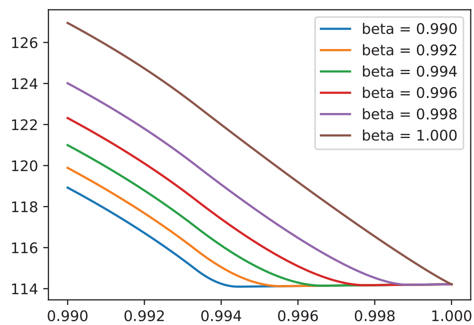

Fig. 4. Conditional value-at-risk for different β-values as a function of B_1 ($B_2 = 1$) when X_1 and X_2 are truncated normal distributed with mean 50 and standard deviation 25.

of B_1 (letting $B_2 = 1$) for different β-values, ranging from 0.990 to 1. The optimal value for B_1 becomes larger and larger as β increases, and when $\beta = 1$, the optimal value for B_1 is 1, as expected.

6. Conclusions and further work

In the present paper, we have seen how multiple reinsurance contracts can be optimized with respect to various types of conditional value-at-risk. In particular, we have introduced the α, β-level conditional value-at-risk, and shown that this can be handled in exactly the same way as the β-level value-at-risk. As β approaches 1, the α, β-level conditional value-at-risk approaches the classical α-level conditional value-at-risk. The optimal solution for this risk measure is obtained as a limit of the optimal solution for the α, β-level conditional value-at-risk. In particular, it is shown that this limit is a stop-loss contract.

Future work in this area includes optimization with respect to other risk measures and objective functions, how to handle dependent risks, as well as a study of how the results may change when the risk distribution parameters are subject to uncertainty.

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