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Environmental contours and time dependence

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Environmental contours are widely used as a basis for e.g., ship design. Such contours are typically used in early design when the strength and failure properties of the object under consideration are not known. An environmental contour describes the tail properties of some relevant environmental variables, and is used as input to the design process. A methodology for constructing environmental contours based on the Rosenblatt transformation has beed used extensively. More recently alternative approach where environmental contours are constructed using Monte Carlo simulation have been developed. Typically, the strength of a structural design is chosen so that the expected return period of a failure event exceeds the desired lifetime of the structure. If time dependence in the environmental variables is neglected, the expected return period is simply the inverse of the failure probability. In a more realistic model, however, such dependence should be included. In this paper we describe a method for constructing an environmental contour where time dependence is taken into account. The method is illustrated with a simple numerical example.

Keywords: Structural reliability, Environmental contour, Structural design, Failure probability, Markov models, Autoregressive models.

1. Introduction

Environmental contours are commonly used as a basis for e.g., ship design, typically in the early design phases when the strength and failure properties of the object under consideration are not fully known. Environmental contours describe tail properties of the distribution of relevant environmental variables. Constructing environmental contours was considered by Winterstein et al. (1993) and Haver and Winterstein (2009). Their procedure starts out by constructing a contour for two independent standard normally distributed variables. This contour is then transformed to a contour in the environmental space using the inverse Rosenblatt transformation. Huseby et al. (2013) noted that the probabilistic properties of the contour is in general not preserved under such transformations, and suggested an alternative approach where environmental contours are constructed directly in the environmental space using Monte Carlo simulation. Improved methods are found in Huseby et al. (2015) and Huseby et al. (2021). A survey of recent development in the field can be found in Ross et al. (2019). For a recent comparison between different methods, see the benchmark study Haselsteiner et al. (2021). In this paper we describe a method for constructing an environmental contour where time dependence is taken into account. For a recent related paper see Mackay et al. (2021).

2. Basic concepts

Let $\boldsymbol{X} = (X_1, X_2) \in \mathcal{X} \subseteq \mathbb{R}^2$ be a vector of environmental variables where e.g.,:

$$X_1 = \text{Wave period}$$
$$X_2 = \text{Significant wave height}$$

An *environmental contour* is defined as the boundary of a set $\mathcal{B} \subseteq \mathcal{X}$, and denoted $\partial\mathcal{B}$. The set $\mathcal{B}$ is called an *environmental contour set*. During the design phase of some structure of interest the environmental contour can be used to identify conditions which the structure should be able to withstand. That is, if $\boldsymbol{X} \in \mathcal{B}$, the structure should function normally. Thus, $\partial\mathcal{B}$ represents the most severe or extreme conditions that the structure should be able to handle.

The *failure region* $\mathcal{F} \subseteq \mathcal{X}$ of a structure is the set of states where the structure fails. See Figure 1. Given an environmental contour set $\mathcal{B}$ the design requirements are satisfied if and only if $\mathcal{F}$ does not intersect with the interior of $\mathcal{B}$. If the set $\mathcal{B}$ is large,

the structure will be subject to strict requirements. As a result, the probability of failure, i.e., the probability that $\boldsymbol{X} \in \mathcal{F}$ is small.

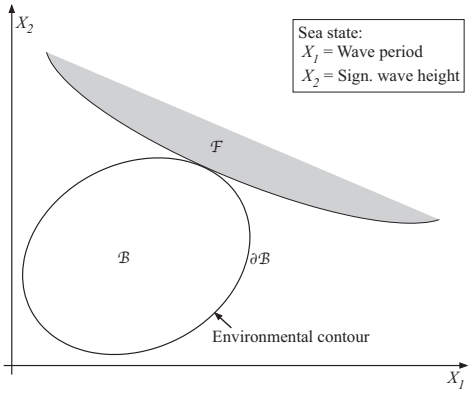

Fig. 1. An environmental contour and a failure region.

In the design phase the exact shape of the failure region $\mathcal{F}$ of a structure is typically unknown. Instead it is assumed to belong to a family of sets denoted by $\mathcal{E}$. The *exceedance probability* of $\mathcal{B}$ with respect to $\mathcal{E}$ is defined as:

$$P_e(\mathcal{B}, \mathcal{E}) = \sup_{\mathcal{F} \in \mathcal{E}} \{P[\boldsymbol{X} \in \mathcal{F}]\}. \tag{1}$$

The exceedance probability is an upper bound on the failure probability of the structure assuming that the true failure region is a member of the family $\mathcal{E}$. A failure region $\mathcal{F} \in \mathcal{E}$ is *maximal* if no region $\mathcal{F}' \in \mathcal{E}$ exists such that $\mathcal{F} \subset \mathcal{F}'$. The family of maximal regions in $\mathcal{E}$ is denoted by $\mathcal{E}^*$. It is easy to see that:

$$P_e(\mathcal{B}, \mathcal{E}) = \sup_{\mathcal{F} \in \mathcal{E}^*} \{P[\boldsymbol{X} \in \mathcal{F}]\}.$$

3. Convex environmental contours

For a given target exceedance probability $p_e \in (0, 0.5)$ a set $\mathcal{B}$ is said to be a *valid* environmental contour set if $P_e(\mathcal{B}, \mathcal{E}) \leq p_e$. In order to simplify the calculations of exceedance probabilities only *convex* contours sets will be considered here. Furthermore, we also assume that the failure regions are *convex*. This implies that all sets in $\mathcal{E}^*$ are half-spaces. This is illustrated in Figure 2.

In order to find a convex contour $\mathcal{B}$ with the desired exceedance probability, we follow Huseby et al. (2015) and let $C(\theta)$ be defined for all angles $\theta \in [0, 2\pi)$ as:

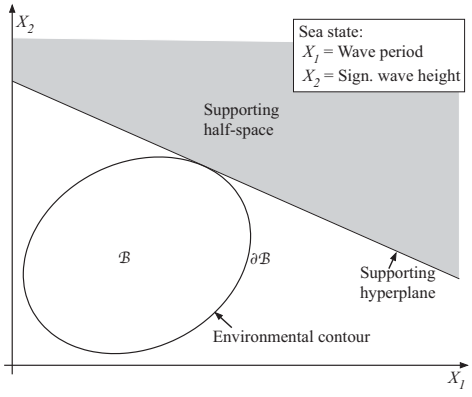

Fig. 2. Supporting hyperplane and half-space

$$C(\theta) = \inf\{y : P[Y(\theta) \geq y] \leq p_e\},$$

where $Y(\theta) = X_1 \cos(\theta) + X_2 \sin(\theta)$ denote the projection of the random sea state vector $\boldsymbol{X}$ onto the unit vector $(\cos(\theta), \sin(\theta))$. The function C is referred to as the p_e-level *percentile function* of the joint distribution of $\boldsymbol{X}$. For $\theta \in [0, 2\pi)$ we also introduce the half-spaces:

$$\mathcal{F}(\theta) = \{\boldsymbol{x} : y(\theta) \geq C(\theta)\}, \tag{2}$$

where $y(\theta) = x_1 \cos(\theta) + x_2 \sin(\theta)$ denote the projection of the point $\boldsymbol{x}$ onto the unit vector $(\cos(\theta), \sin(\theta))$. We then assume that we can find a convex contour set $\mathcal{B}$ such that the family of maximal failure regions is:

$$\mathcal{E}^* = \{\mathcal{F}(\theta) : \theta \in [0, 2\pi)\}$$

Then it follows that:

$$\begin{aligned} P_e(\mathcal{B}, \mathcal{E}) &= \sup_{\mathcal{F} \in \mathcal{E}^*} \{P[\boldsymbol{X} \in \mathcal{F}]\} \\ &= \sup_{\theta \in [0, 2\pi)} \{P[\boldsymbol{X} \in \mathcal{F}(\theta)]\} \\ &= \sup_{\theta \in [0, 2\pi)} \{P[Y(\theta) \geq C(\theta)]\} = p_e \end{aligned}$$

Thus, $\mathcal{B}$ is a valid environmental contour set. In fact $\mathcal{B}$ is the minimal set with this property. By using Monte Carlo simulation the function $C(\theta)$ can be estimated, and hence also the resulting

environmental contour set $\mathcal{B}$. Note that in this case the environmental contour set is given by:

$$\mathcal{B} = \bigcap_{\theta \in [0,2\pi)} \bar{\mathcal{F}}(\theta), \tag{3}$$

where $\bar{\mathcal{F}}(\theta) = \{\boldsymbol{x} : y(\theta) \leq C(\theta)\}$ for all $\theta \in [0, 2\pi)$. See Huseby et al. (2015) and Huseby et al. (2021) for further details.

4. Return periods and dependence

In many practical situations it is of interest to determine the *expected return period* of a given failure event. Assuming a simple *discrete time* model with no time dependence, the expected return period is essentially the inverse of the exceedance probability. Thus, for a given expected return period, we can simply calculate the corresponding exceedance probability, and then construct the resulting environmental contour. The main objective of this paper, however, is to study the connection between the return period and the exceedance probability in cases with time dependence. Thus, we consider the state of the environment which is observed at discrete points of time $0 = t_0 < t_1 < t_2 < \cdots$. We let $\boldsymbol{X}_i = (X_{i1}, X_{i2})$ denote the observation at time t_i, $i = 0, 1, 2, \ldots$. The discrete time stochastic process $\{\boldsymbol{X}_i\}$ will be called the *environmental process*. For each $\theta \in [0, 2\pi)$ we also let:

$$Z_i(\theta) = \mathrm{I}(\boldsymbol{X}_i \in \mathcal{F}(\theta)), \quad i = 0, 1, 2, \ldots \tag{4}$$

where $\mathrm{I}(\cdot)$ denotes the indicator function. We assume that $\{Z_i(\theta)\}$, at least approximately, can be modeled as a Markov process with state space $\mathcal{S} = \{0, 1\}$, and transition probability matrix $\boldsymbol{P}(\theta)$ given by:

$$\boldsymbol{P}(\theta) = \begin{bmatrix} p_{00}(\theta), p_{01}(\theta) \\ p_{10}(\theta), p_{11}(\theta) \end{bmatrix}$$

We also introduce the stationary distribution of the process, given by:

$$\pi_0(\theta) = P(Z_i(\theta) = 0),$$
$$\pi_1(\theta) = P(Z_i(\theta) = 1),$$

for $i = 1, 2, \ldots$. The stationary distribution can be found using standard methods for discrete time discrete space Markov processes, by solving the equations:

$$\boldsymbol{\pi}(\theta)\boldsymbol{P}(\theta) = \boldsymbol{\pi}(\theta), \text{ and } \pi_0(\theta) + \pi_1(\theta) = 1,$$

where $\boldsymbol{\pi}(\theta) = (\pi_0(\theta), \pi_1(\theta))$. As a result we get:

$$\pi_0(\theta) = \frac{p_{10}(\theta)}{p_{10}(\theta) + p_{01}(\theta)},$$
$$\pi_1(\theta) = \frac{p_{01}(\theta)}{p_{10}(\theta) + p_{01}(\theta)}.$$

At the same time we have:

$$P(Z_i(\theta) = 1) = P(\boldsymbol{X}_i \in \mathcal{F}(\theta)) = p_e$$

Combining this, we get that:

$$\pi_1(\theta) = \frac{p_{01}(\theta)}{p_{10}(\theta) + p_{01}(\theta)} = p_e. \tag{5}$$

Assuming that $Z_0(\theta) = 0$, we consider the point of time where the process $\{Z_i(\theta)\}$ first enter state 1, denoted:

$$N(\theta) = \min\{n \geq 1 : Z_n(\theta) = 1\}. \tag{6}$$

Note that due to the relation between the processes $\{Z_i(\theta)\}$ and $\{\boldsymbol{X}_i\}$, Eq. (6) can also be expressed as:

$$N(\theta) = \min\{n \geq 1 : \boldsymbol{X}_n \in \mathcal{F}(\theta)\}. \tag{7}$$

The random variable $N(\theta)$ is referred to as the *return period* of the failure event $\{\boldsymbol{X}_n \in \mathcal{F}(\theta)\}$. The *expected return period*, $\mathrm{E}[N(\theta)]$ can easily be determined by conditioning on $Z_1(\theta)$ and using that $Z_0(\theta) = 0$, implying that:

$$\mathrm{E}[N(\theta)] = p_{00}(\theta) \cdot (1 + \mathrm{E}[N(\theta)]) + p_{01}(\theta) \cdot 1$$

Solving this equation with respect to $\mathrm{E}[N(\theta)]$, we get that:

$$\mathrm{E}[N(\theta)] = \frac{1}{p_{01}(\theta)} \tag{8}$$

By Eq. (5) we have that:

$$p_{01}(\theta) = p_e(p_{10}(\theta) + p_{01}(\theta))$$

Inserting this into Eq. (8) we get:

$$\mathrm{E}[N(\theta)] = \frac{1}{p_e(p_{01}(\theta) + p_{10}(\theta))} \tag{9}$$

In the special case where $(p_{01}(\theta) + p_{10}(\theta)) = 1$, the expected return period simplifies to:

$$\mathrm{E}[N(\theta)] = \frac{1}{p_e}, \quad \text{for all } \theta \in [0, 2\pi). \tag{10}$$

Since the row sums in the transition probability matrix $\boldsymbol{P}(\theta)$ are one, $(p_{01}(\theta) + p_{10}(\theta)) = 1$ also implies that:

$$p_{10}(\theta) = 1 - p_{01}(\theta) = p_{00}(\theta)$$
$$p_{11}(\theta) = 1 - p_{10}(\theta) = p_{01}(\theta)$$

Hence, the transition probability matrix becomes:

$$\boldsymbol{P}(\theta) = \begin{bmatrix} p_{00}(\theta), p_{01}(\theta) \\ p_{00}(\theta), p_{01}(\theta) \end{bmatrix}$$

Since the rows of $\boldsymbol{P}(\theta)$ are equal, this corresponds to a situation where $Z_0(\theta), Z_1(\theta), Z_2(\theta), \ldots$ are *independent* of each other.

In many cases, however, it may be more realistic to assume that $Z_0(\theta), Z_1(\theta), Z_2(\theta), \ldots$ are *positively correlated*, which corresponds to the following inequalities:

$$p_{01}(\theta) < p_{11}(\theta), \tag{11}$$
$$p_{10}(\theta) < p_{00}(\theta) \tag{12}$$

The condition Eq. (11) means that the probability that $\boldsymbol{X}_{i+1} \in \mathcal{F}(\theta)$ given that $\boldsymbol{X}_i \in \mathcal{F}(\theta)$ is greater than the probability that $\boldsymbol{X}_{i+1} \in \mathcal{F}(\theta)$ given that $\boldsymbol{X}_i \notin \mathcal{F}(\theta)$. That is, a failure event is more likely to happen if the process is already in a failed state, than if it is not in such a state. The condition Eq. (12) has a similar interpretation.

Using again that the row sums in the transition probability matrix $\boldsymbol{P}(\theta)$ are one, it is easy to see that both Eq. (11) and Eq. (12) are equivalent to the condition that $p_{01}(\theta) + p_{10}(\theta) < 1$. Thus, if $Z_0(\theta), Z_1(\theta), Z_2(\theta), \ldots$ are positively correlated, this implies that:

$$\mathrm{E}[N(\theta)] = \frac{1}{p_e(p_{01}(\theta) + p_{10}(\theta))} > \frac{1}{p_e}$$

We now assume that instead of specifying a fixed target exceedance probability p_e, we specify a fixed *target expected return period*, denoted μ_e. The goal is then to find an adjusted target exceedance probability $\tilde{p}_e > p_e$ such that:

$$\frac{1}{\tilde{p}_e(p_{01}(\theta) + p_{10}(\theta))} \geq \mu_e \text{ for all } \theta \in [0, 2\pi)$$

In order to find $\tilde{p}_e$ we let:

$$\theta^* = \underset{\theta \in [0, 2\pi)}{\operatorname{argmax}} [p_{01}(\theta) + p_{10}(\theta)].$$

A sufficient condition for $\mathrm{E}[N(\theta)] \geq \mu_e$ for all $\theta \in [0, 2\pi)$ is then that:

$$\mathrm{E}[N(\theta^*)] = \frac{1}{\tilde{p}_e(p_{01}(\theta^*) + p_{10}(\theta^*))} = \mu_e$$

Hence, the target exceedance probability can be adjusted to:

$$\tilde{p}_e = \frac{1}{\mu_e(p_{01}(\theta^*) + p_{10}(\theta^*))}$$

Based on the adjusted target exceedance probability, we can calculate an adjusted percentile function $\tilde{C}(\theta)$ satisfying:

$$P(Y(\theta) \geq \tilde{C}(\theta)) = \tilde{p}_e$$

and use this to determine an adjusted environmental contour set $\tilde{\mathcal{B}}$ with an expected return period which is closer to the target value μ_e.

Note that if the exceedance probability is increased from p_e to $\tilde{p}_e$ and this also affects the transitions probabilities of the processes $\{Z_i(\theta)\}$. In fact, this typically increases the correlations. Thus, $(p_{01}(\theta) + p_{10}(\theta))$ becomes smaller for all $\theta \in [0, 2\pi)$, and as a result the return periods may still be a bit longer than the target value. Since longer return periods are desirable compared to shorter periods, this potential error is considered to be *conservative*.

5. Modeling time dependence

In this section we describe a model for the environmental process $\{\boldsymbol{X}_i\}$ which includes time dependence assuming that:

$$\boldsymbol{X}_i = \Psi^{-1}(\boldsymbol{W}_i), \quad i = 0, 1, 2, \ldots, \tag{13}$$

where $\boldsymbol{W}_i = (W_{i1}, W_{i2})$ has a standard bivariate normal distribution, $i = 0, 1, 2, \ldots$, and Ψ^{-1} denotes the inverse of the well-known *Rosenblatt transformation*, chosen so that $\boldsymbol{X}_i$ gets the desired bivariate distribution $i = 0, 1, 2, \ldots$.

In order to ensure that $\{\boldsymbol{X}_i\}$ (and hence, also $\{Z_i\}$) is a Markov process, we let $\{\boldsymbol{W}_i\}$ be a

bivariate stationary AR(1) process. That is, for $j = 1, 2$ we let:

$$W_{ij} = \rho_j W_{i-1,j} + \bar{\rho}_j V_{ij}, \quad i = 1, 2, \ldots,$$

where $\boldsymbol{W}_0, \boldsymbol{V}_1, \boldsymbol{V}_2, \ldots$ is a sequence of independent standard bivariate normally distributed vectors, and $\bar{\rho}_j = \sqrt{1-\rho_j^2}$, $j = 1, 2$. We then have for $i = 1, 2, \ldots$ and $j = 1, 2$:

$$\mathrm{E}[W_{i,j}] = \rho_j \,\mathrm{E}[W_{i-1,j}] + \bar{\rho}_j \,\mathrm{E}[V_{i,j}] = 0$$

$$\begin{aligned}\mathrm{Var}(W_{i,j}) &= \rho_j^2 \,\mathrm{Var}(W_{i-1,j}) + \bar{\rho}_j^2 \,\mathrm{Var}(V_{i,j}) \\ &= \rho_j^2 + (1-\rho_j^2) = 1\end{aligned}$$

$$\begin{aligned}&\mathrm{Cov}(W_{i,j}, W_{i-1,j}) \\ &\quad = \mathrm{Cov}(\rho_j W_{i-1,j} + \bar{\rho}_j V_{i,j}, W_{i-1,j}) \\ &\quad = \rho_j \,\mathrm{Cov}(W_{i-1,j}, W_{i-1,j}) \\ &\quad + \bar{\rho}_j \,\mathrm{Cov}(V_{i,j}, W_{i-1,j}) \\ &\quad = \rho_j \,\mathrm{Var}(W_{i-1,j}) + \bar{\rho}_j \cdot 0 = \rho_j\end{aligned}$$

Thus, it follows that $\boldsymbol{W}_0, \boldsymbol{W}_1, \boldsymbol{W}_2, \ldots$ are standard bivariate normally distributed variables with correlations ρ_1 and ρ_2 for component 1 and 2 respectively. Hence, $\{\boldsymbol{X}_i\}$ also gets the desired marginal distribution.

6. Estimating the transition probability matrices

We now assume that we are given a convex environmental contour set $\mathcal{B}$, which is estimated assuming *no time dependence*, and using a target exceedance probability p_e. We then choose $\theta \in [0, 2\pi)$, and consider the process $\{Z_i(\theta)\}$ defined relative to $\mathcal{B}$ by Eq. (4). That is, the failure region $\mathcal{F}(\theta)$ is a supporting half-space of the given set $\mathcal{B}$.

The transition probability matrices $\boldsymbol{P}(\theta)$ can be estimated by running a Monte Carlo simulation of the process $\{\boldsymbol{X}_i\}$. This is easily done by generating $\boldsymbol{W}_0, \boldsymbol{V}_1, \boldsymbol{V}_2, \ldots, \boldsymbol{V}_N$ as a sequence of independent standard bivariate normally distributed vectors, and then use the transformations described above. As a result we get a sequence:

$$\boldsymbol{X}_0, \boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_N$$

sampled from the environmental process.

To estimate the transition probability matrix $\boldsymbol{P}(\theta)$, we need to count the *number of transitions* made by the process $\{Z_i(\theta)\}$. Thus, we let:

$$\begin{aligned}S_{00}(\theta) &= \sum_{i=1}^{N} \mathrm{I}(Z_{i-1}(\theta) = 0, Z_i(\theta) = 0) \\ S_{01}(\theta) &= \sum_{i=1}^{N} \mathrm{I}(Z_{i-1}(\theta) = 0, Z_i(\theta) = 1) \\ S_{10}(\theta) &= \sum_{i=1}^{N} \mathrm{I}(Z_{i-1}(\theta) = 1, Z_i(\theta) = 0) \\ S_{11}(\theta) &= \sum_{i=1}^{N} \mathrm{I}(Z_{i-1}(\theta) = 1, Z_i(\theta) = 1)\end{aligned}$$

Thus, $S_{st}(\theta)$ is the number of transitions from state s to state t made by the process $\{Z_i(\theta)\}$ during the simulation, $s, t = 0, 1$. We then obtain the following unbiased estimates of the transition probabilities:

$$\begin{aligned}\hat{p}_{00}(\theta) &= \frac{S_{00}(\theta)}{S_{00}(\theta) + S_{01}(\theta)} \\ \hat{p}_{01}(\theta) &= \frac{S_{01}(\theta)}{S_{00}(\theta) + S_{01}(\theta)} \\ \hat{p}_{10}(\theta) &= \frac{S_{10}(\theta)}{S_{10}(\theta) + S_{11}(\theta)} \\ \hat{p}_{11}(\theta) &= \frac{S_{11}(\theta)}{S_{10}(\theta) + S_{11}(\theta)}\end{aligned}$$

The matrix $\boldsymbol{P}(\theta)$ will be estimated for a suitable set of values $\theta_1, \ldots, \theta_k$, evenly spread out in the interval $[0, 2\pi)$.

The above Monte Carlo method is very easy to implement. Unfortunately, however, it does not produce very precise results. The problem is that since the exceedance probability p_e is typically very small, most of the time the $\{Z_i(\theta)\}$-processes will be in state 0. Thus, even if N is large, the number of transitions to state 1 will typically be relatively small. In order to improve the results, we will use a technique similar to the one used when estimating the percentile function $C(\theta)$ as part of the estimation of the safe $\mathcal{B}$.

In order to explain the idea, we let $r > 0$, and define $\mathcal{O}_r$ as the set of points within a circle with radius r centered at the origin. That is:

$$\mathcal{O}_r = \{\boldsymbol{w} = (w_1, w_2) : w_1^2 + w_2^2 \leq r^2\}$$

We then assume that the radius r is chosen so that:

$$\Psi^{-1}(\mathcal{O}_r) \;=\; \{\boldsymbol{x} = \Psi^{-1}(\boldsymbol{w}) : \boldsymbol{w} \in \mathcal{O}\} \;\subseteq\; \mathcal{B},$$

where as before Ψ^{-1} denotes the inverse Rosenblatt transformation. From this it follows that if $\boldsymbol{W}_i \in \mathcal{O}_r$, then $\boldsymbol{X}_i = \Psi^{-1}(\boldsymbol{W}) \in \mathcal{B}$. Moreover, by Eq. (3) it follows that if $\boldsymbol{W}_i \in \mathcal{O}_r$, then for all $\theta \in [0, 2\pi)$ we have:

$$Y_i(\theta) = X_{i1}\cos(\theta) + X_{i2}\sin(\theta) \leq C(\theta).$$

Finally, since $Z_i(\theta) = \mathrm{I}(Y_i(\theta) > C(\theta))$ it follows that if $\boldsymbol{W}_i \in \mathcal{O}_r$, then for all $\theta \in [0, 2\pi)$ we have:

$$Z_i(\theta) = 0.$$

Thus, as long as the $\{\boldsymbol{W}_i\}$-process is within the set $\mathcal{O}_r$, we do not need to know the exact value of the process to determine the state of the $\{Z_i(\theta)\}$-process.

In order to utilize this observation, we introduce:

$$\{j : \boldsymbol{W}_j \notin \mathcal{O}_r\} = \{j_1, j_2, \ldots\}.$$

Thus, the process $\boldsymbol{W}_j$ is outside of the set $\mathcal{O}_r$ for $j = j_1, j_2, \ldots$. Whenever this happens, we compute the corresponding value of $\boldsymbol{X}$ along with the index j. Using this procedure we get the following data:

$$\boldsymbol{X}_i = \Psi^{-1}(\boldsymbol{W}_{j_i}), \quad I_i = j_i, \quad i = 1, 2, \ldots, N$$

This implies that all the cases where $\boldsymbol{W}_j$ are inside the set $\mathcal{O}_r$ are discarded. Still, since we store the indices of the values of $\boldsymbol{W}_j$ which are outside the set $\mathcal{O}_r$ in $\{I_i\}$, we keep track of the number of discarded results and where in the sequence they occur. In fact, given the results $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N$ and $I_1, \ldots, I_N$, we can compute the number of transitions of the $\{Z_i(\theta)\}$-processes, and thus, estimate the transition probability matrices $\boldsymbol{P}(\theta)$. In order to explain this in more detail, we choose $\theta \in [0, 2\pi)$, and consider transitions of the process $\{Z_i(\theta)\}$ between $\boldsymbol{X}_{i-1}$ and $\boldsymbol{X}_i$, noting that the total number of transitions in this interval is given by the index difference $(I_i - I_{i-1})$.

CASE 1. $Y_{i-1}(\theta) \leq C(\theta)$ and $Y_i(\theta) \leq C(\theta)$.
In this case the process $\{Z_i(\theta)\}$ is in state 0 all the time. Thus, all the $(I_i - I_{i-1})$ transitions are from state 0 to state 0.

CASE 2. $Y_{i-1}(\theta) \leq C(\theta)$ and $Y_i(\theta) > C(\theta)$.
In this case the process $\{Z_i(\theta)\}$ is in state 0 all the time except for the last point of time where it enters state 1. Thus, the $(I_i - I_{i-1} - 1)$ first transitions are from state 0 to state 0, while the last transition is from state 0 to state 1.

CASE 3. $Y_{i-1}(\theta) > C(\theta)$ and $Y_i(\theta) \leq C(\theta)$.
In this case the process $\{Z_i(\theta)\}$ starts out in state 1 and then immediately enters state 0. Thus, the first transition is from state 1 to state 0, while the $(I_i - I_{i-1} - 1)$ transitions are from state 0 to state 0.

CASE 4. $Y_{i-1}(\theta) > C(\theta)$ and $Y_i(\theta) > C(\theta)$.
In this case we need to distinguish between two possibilities. If $I_i - I_{i-1} = 1$, the process $\{Z_i(\theta)\}$ is in state 1. Thus, there is exactly 1 transition, and this is from state 1 to state 1. On the other hand, if $I_i - I_{i-1} > 1$, then the first transition is from state 1 to state 0, while the last transition is from state 0 to state 1. The remaining $(I_i - I_{i-1} - 2)$ transitions are from state 0 to state 0.

Using these four cases all transitions between $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$, all transitions between $\boldsymbol{X}_2$ and $\boldsymbol{X}_3$, up to all transitions between $\boldsymbol{X}_{N-1}$ and $\boldsymbol{X}_N$, can be counted. Thus, the quantities $S_{00}(\theta), S_{01}(\theta), S_{10}(\theta), S_{11}(\theta)$ can be calculated, and hence, the resulting transition probabilities can be estimated using the unbiased estimators $\hat{p}_{00}(\theta), \hat{p}_{01}(\theta), \hat{p}_{10}(\theta), \hat{p}_{11}(\theta)$.

Using these transition probability estimates we can also estimate the expected return period $\mathrm{E}[N(\theta)]$ as a function of the angle θ:

$$\widehat{\mathrm{E}[N(\theta)]} = \frac{1}{p_e(\hat{p}_{01}(\theta) + \hat{p}_{10}(\theta))}$$

Finally, the adjusted target exceedance probability, $\tilde{p}_e$ can be estimated by:

$$\hat{\tilde{p}}_e = \frac{1}{\mu_e(\hat{p}_{01}(\theta^*) + \hat{p}_{10}(\theta^*))}$$

where:

$$\theta^* = \underset{\theta \in [0, 2\pi)}{\mathrm{argmax}}[\hat{p}_{01}(\theta) + \hat{p}_{10}(\theta))]$$

Based on the estimated adjusted target exceedance probability, $\hat{\tilde{p}}_e$, we can then estimate an adjusted environmental contour set $\hat{\tilde{\mathcal{B}}}$ assuming no time dependence.

Note, that in order to generate N data points, we need to generate a much higher number of $\boldsymbol{W}_j$-values. However, the estimates become *far more precise*. Moreover, the estimation is *much faster* since the calculations are done based on just N data points compared to the full set of data points.

7. A numerical example

We now illustrate the method presented in the previous section by an example where we let $p_e = 2.74 \cdot 10^{-4}$. Given a sample rate of 1 per day and no time dependence, this corresponds to a target expected return period of $\mu_e = 10$ years. For the AR(1) process we let $\rho_1 = \rho_2 = 0.95$.

The example represents total sea wave data from West Shetland. A three-parameter Weibull distribution is used for the significant wave height, H, while a lognormal conditional distribution is used for the wave period, T. The Weibull distribution is parameterized by a location parameter, γ, a scale parameter α, and a shape parameter β. The lognormal distribution has two parameters, the log-mean μ and the log-standard deviation σ. The dependence between H and T is modeled by letting the parameters μ and σ be expressed in terms of H:

$$\mu = \mathrm{E}[\ln(T)|H = h] = a_1 + a_2 h^{a_3},$$
$$\sigma = SD[\ln(T)|H = h] = b_1 + b_2 e^{b_3 h}.$$

The distribution parameters, estimated using data from West Shetland, are listed in Table 1 and 2. We start out by estimating an environmental contour set $\mathcal{B}$ assuming no time dependence and with target exceedance probability p_e. We then proceed by running a Monte Carlo simulation with the specified time dependence, and estimate the resulting transition probabilities. In Figure 3 we have plotted the estimated expected return period, $\mathrm{E}[N(\theta)]$ as a function of the angle θ (in degrees). Due to simulation uncertainty the curve appears to be very unstable. However, the actual variation is not very significant. The shortest expected return period is estimated to be 21.7 years which is more than *twice* the target value of 10 years. The resulting adjusted target exceedance probability is the $\tilde{p} = 5.94 \cdot 10^{-4}$.

Table 1. Fitted parameter for the three-parameter Weibull distribution

α	β	γ
2.259	1.285	0.701

Table 2. Fitted parameter for the conditional log-normal distribution

	$i = 1$	$i = 2$	$i = 3$
a_i	1.069	0.898	0.243
b_i	0.025	0.263	-0.148

Based on the adjusted target exceedance probability an adjusted environmental contour set $\tilde{\mathcal{B}}$ is estimated. Based on this set we then run another Monte Carlo simulation with the specified time dependence, and estimate the resulting transition probabilities. In Figure 4 we have plotted the resulting estimated expected return period for $\tilde{\mathcal{B}}$. The shortest expected return period is estimated to be 10.4 years which is much closer to the target value of 10 years.

In Figure 5 we have plotted both the original contour (the outer curve) and the adjusted contour (the inner curve). We see that due to the time dependence the environmental contour set can be reduced significantly.

In principle one could compute a new adjusted target exceedance probability, and yet another adjusted environmental contour set and so on. By repeating this process, the resulting expected return period should ideally converge towards the target value. However, it turns out that the effect on the environmental contour is very limited.

8. Conclusions

We have presented a method for adjusting environmental contours when the environmental variables are subject to time dependence. The time dependence is modeled by a simple two-dimensional AR(1) process. The environmental contours are constructed using a direct Monte Carlo simulation

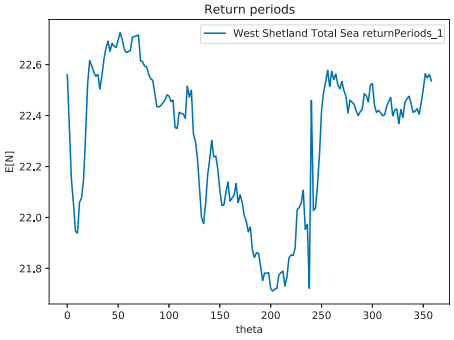

Fig. 3. Expected return periods as a function of θ

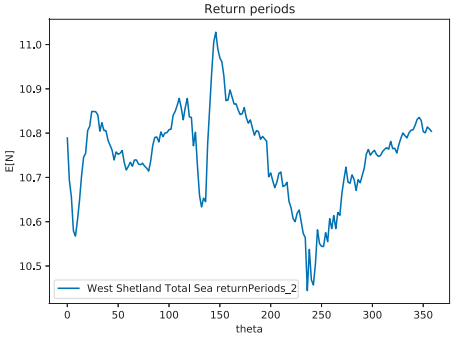

Fig. 4. Adjusted return period as a function of the angle θ

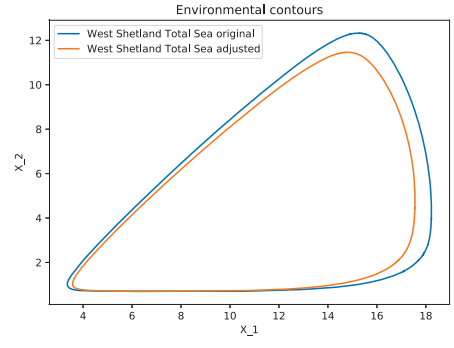

Fig. 5. Original and adjusted contours

approach with importance sampling. The transition probabilities are estimated using rejection sampling. This yields greatly improved precision while still enabling efficient calculations. The numerical example shows that time dependence can have significant impact on return periods where the adjusted environmental contour set becomes a subset of the original set. Note that when using an AR(1) process, the time correlation needs to be quite large in order to obtain a realistic model. In an upcoming study more advanced time series models will be investigated, and we will also study how to fit such model to real data.

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