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A phase-type maintenance model considering condition-based inspections and delays before the repairs

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Markov models are widely used in maintenance modelling and system performance analysis due to their computational efficiency and analytical traceability. However, these models are usually restricted by the use of exponential distributions, which are the base of the Markov modelling. Phase-type distributions provide a tool to approximate an adequate distribution, such as Weibull, log-normal and so on, by means of Markov processes.

Our earlier work proposes a phase-type maintenance model considering both condition-based inspections and delays before the repairs, where extra matrices are defined in the modelling of repair delays to keep track of the probability masses to repair. The model provides quite good estimations but is complex and requires good knowledge in its implementation.

This paper aims to get rid of the extra matrices and investigate the modelling of the repair delays with phase-type distributions. An illustration case of road bridges is presented to demonstrate the modelling process and the results.

Keywords: Maintenance modelling, Phase-type distribution, Condition-based inspections, Delays before repairs.

1. Introduction

With the rapid development of monitoring technologies, condition-based maintenance (CBM) is gaining popularity in the past decades. Various maintenance management systems have been developed to store the monitoring results and assist in planning different maintenance tasks. These tasks are often planned based on predefined rules derived from the analysis results of maintenance models, among which Markovian models are used extensively due to their computational efficiency and analytical traceability.

The Markov models in the previous literature are usually restricted by the memory-less property of the Markov Chain, which is shown as exponential sojourn times between the discrete states. Phase-type (PH) distributions provide a solution to fit general distributions by means of Markov chains and thus capture the non-Markovian deterioration while still taking advantage of the tractability of the Markovian models. The class of PH distributions has strong versatility in approximating a generic distribution, and any non-negative distribution can be approximated arbitrarily close by a PH distribution (Lindqvist and Kjølen, 2018).

In our earlier work, a Markov-based maintenance model is proposed considering both condition-based inspections and deterministic delays before the repairs (see Sun and Vatn (2023)). The deterioration process is approximated with PH distributions, while the delays before the repairs are modelled by defining extra matrices and keeping track of the probability masses to repair. The model provides quite good estimations but is complex and requires good knowledge in its implementation. In addition, it requires some modification if we want to model the repair delays as a distribution instead of deterministic values.

This paper considered the same case of bridge management in Norway and investigated the modelling of both the bridge deterioration and the repair delays with phase-type distributions. The delays are assumed to be lognormal-distributed instead of deterministic to limit the number of phases required.

The remainder of this paper is organised as follows: Section 2 summarised the background knowledge in phase-type distribution and the fitting approach used in this paper. The detailed modelling process is then presented in Section 3, followed by some numerical results in Section 4. We end with a summary in Section 5. The notations used in this paper are listed in Table 1.

Table 1. Notations

Notation	Interpretation
$\boldsymbol{\alpha}_j$	Initial probability vector for a phase-type distribution
$\mathbf{S}_j$	Transition matrix among the transient states for a phase-type distribution
τ_j	Intervals between inspections for the bridge in state j. It is assumed that $\tau_j = k_j \tau_n$
$\mathbb{S}_d$	A set of Markov states describing the bridge deterioration
$\mathbb{S}_r$	A set of Markov states describing the repair process
β_u^j	Initial probability for the uth phase of $\mathbb{S}_r$ for vector $\mathbf{P}^j$
μ_u^j	Transition rate from the uth phase of $\mathbb{S}_r$ for vector $\mathbf{P}^j$
$P_i^{j,l_j}(t)$	Probability of the bridge being in macro state i at time t while following an inspection regime $\{j, l_j\}$
$\boldsymbol{A}^j$	Full transition matrix for $\mathbf{P}^{j,l_j}$ vectors
$\boldsymbol{A}_d$	Transition matrix among $\mathbb{S}_d$
$\boldsymbol{A}_{r,u}^j$	Transition matrix from the uth phase to the $u+1$th phase of $\mathbb{S}_r$ in $\boldsymbol{A}^j$
$\boldsymbol{B}^j$	Full inspection matrix for $\mathbf{P}^{j,l_j}$ vectors
$\boldsymbol{B}_u^j$	Transition rates from macro state j to the uth phase of $\mathbb{S}_r$ in $\boldsymbol{B}^j$
e_S, e_M, e_L	Probability that a minor repair, a major repair or rehabilitation successfully restores the bridge to the desired state
ϕ^j	Intended state change before and after the repair for the bridge in state j
θ	A maintenance action (inspection or a type of repair)
Θ	A set of all possible actions
C_θ	Cost per action θ
d_f	Expected duration of being in state 5
C_f	Penalty per unit time when the bridge is in state 5

2. Phase-type distributions

Consider a Markov process with state space $\mathcal{S} = \{1, 2, ..., m, m+1\}$, where the state $1, 2, ..., m$ are transient and state $m+1$ is absorbing. The time to absorption is then said to follow a phase-type distribution. The infinitesimal generator matrix $\mathbf{A}$ is a $(m+1) \times (m+1)$ matrix given by:

$$\mathbf{A} = \begin{pmatrix} \boldsymbol{S} & \boldsymbol{s} \\ \mathbf{0} & 0 \end{pmatrix} \tag{1}$$

Where $\mathbf{S}$ is a $m \times m$ matrix defining the transition rates among the transient states; $\mathbf{s}$ is a $m \times 1$ vector describing the transition rates from the transient states to the absorbing state; $\mathbf{0}$ is a $1 \times m$ vector of zeros.

An essential task in the implementation of PH distributions is to determine its representation ($\boldsymbol{\alpha}$, $\boldsymbol{S}$) based on either empirical data or probability density functions. Many fitting approaches are available in the literature, generally classified as moment-based and likelihood-based. In this process, the number of phases m and the structure of the transient matrix $\boldsymbol{S}$ need to be determined.

Estimating all transition probabilities can be computationally expensive for a full PH distribution. Therefore, some special cases are usually adopted to limit the model complexity. A widely used subclass is the Acyclic PH (APH) distribution, where the $\boldsymbol{S}$ matrix becomes an upper triangular matrix, can always be represented in three canonical forms, and the parameter estimation essentially reduces to PH fitting with these canonical forms (Okamura and Dohi, 2016). This paper uses the moment-based fitting function in the Butools toolbox (BuTools, 2015) to derive the parameters for the PH distribution.

2.1. *PH expansion of multi-state Markov models*

A multi-state Markov model can be extended to its PH expansion by approximating the sojourn times at each state with PH distributions and merging them together. In the merging, the transition rates to the absorbing state, i.e. the next macro state, are split according to the initial probability vector of the next states to ensure a good fitting (see proof in Laskowska and Vatn (2020)). Suppose there is a Markov chain with n macro states, where the sojourn times at state $j \in \{1, 2, ...n-1\}$ are approximated with PH distributions of m_j phases. Fig.1 visualizes the concept of merging.

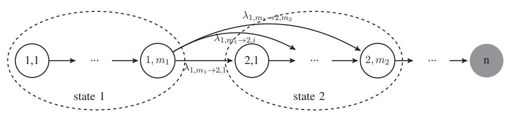

Fig. 1. PH expansion of multi-state Markov model

The macro state $j+1$ can be regarded as the absorbing state from the intermediate state j, m_j, where the transition rates can be derived by

$$[\lambda_{j.m_j \to \{j+1\},1}, ..., \lambda_{j,m_j \to \{j+1\},m_{j+1}}] = \boldsymbol{\alpha}_{j+1} \cdot \lambda_{j \to j+1} \tag{2}$$

The expansion of the multi-state Markov model can be represented with initial probability vector $\boldsymbol{P}_0 = [\boldsymbol{\alpha}_1, \mathbf{0}]$ and transition matrix $\boldsymbol{A}$.

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{S}_1 & \boldsymbol{S}_{1\to 2} & & & \\ & \boldsymbol{S}_2 & \boldsymbol{S}_{2\to 3} & & \\ & & \ddots & & \\ & & & \boldsymbol{S}_{n-1} & \boldsymbol{S}_{n-1\to n} \\ & & & & 0 \end{pmatrix}, \quad \boldsymbol{S}_{j\to j+1} = \begin{pmatrix} \mathbf{0} \\ \vdots \\ \boldsymbol{\alpha}_{j+1} \cdot \lambda_{j,m_j \to j+1} \end{pmatrix} \tag{3}$$

According to the Chapman-Kolmogorov differential equation, the probability distribution at time t can be derived by:

$$\mathbf{P}(t) = \mathbf{P}_0 \cdot e^{\mathbf{A}t} \tag{4}$$

3. Model Description

3.1. *Model assumptions*

The following assumptions are adopted in this paper.

(i) The bridge deterioration is described by five macro discrete condition states. The sojourn times for bridge deterioration are assumed Weibull-distributed. Based on the parameters found in Fang and Sun (2019), three states are required for approximating the sojourn times in state 1 and state 2, while two states are enough for state 3 and state 4.

(ii) The bridge condition can be revealed by condition-based inspections and after each repair. The time for the next inspection is based on the bridge condition revealed at the current inspection. Let t_0 be the current inspection time, and j be the bridge condition revealed from an inspection; the time for the next inspection would be at $t_0 + \tau_j$, where $j \in \{1, 2, 3, 4, 5\}$ denotes the bridge state found during the inspection. It is assumed that $\tau_j = k_j \tau_5$, where k_j are all integers for simplification.

(iii) All inspections are perfect and can reveal the actual bridge condition.

(iv) There are significant waiting times before the conduction of repairs. The waiting times depend on the bridge's condition revealed during an inspection and are assumed to be lognormal-distributed, which is a distribution commonly used for modelling repair times (Rausand et al., 2021). Extra penalties will be triggered when the bridge is in state 5.

(v) There are three levels of repairs: minor repair (MiRep), major repair (MaRep) and rehabilitation (Rehab). The repairs will most restore the bridge to the desired state but may fail to achieve the planned bridge improvement in some cases. Table 2 shows the ideal bridge improvement and success probability of different levels of repairs.

Table 2. Different levels of repairs

Level of repair	States improvement		Success
	Success	Failure	probability
MiRep	1	0	e_S
MaRep	2	1	e_M
Rehab	3	2	e_L

Note: State improvement = 1 means to improve the bridge condition by 1, e.g. from state 4 to state 3.

(vi) The bridge can further deteriorate while waiting for the repair. In this case, the bridge will follow the earlier time for repair between the original and the rescheduled one.

3.2. *Modelling of condition-based inspections*

This section is based on Sun and Vatn (2023). To represent various inspection regimes, a total of k_1 $\mathbf{P}^1$ vectors, k_2 $\mathbf{P}^2$ vectors, k_3 $\mathbf{P}^3$ vectors, k_4 $\mathbf{P}^4$ vectors, and one $\mathbf{P}^5$ vector are defined. Each vector represents an inspection cycle with a unique starting point and inspection interval. The change of inspection regimes is represented by shuffling the probability mass between these $\mathbf{P}$ vectors. This process is illustrated in Fig. 2. Consider an inspection at $t = k_1\tau_5$, for vector $\mathbf{P}^{j,l_j}$, the probability mass in macro state $i \neq j$ will be moved to a $\mathbf{P}^i$ vector where the time for the next inspection is $k_1\tau_5 + \tau_i$.

3.3. *Markov processes for the* $\mathbf{P}$ *vectors*

The above-mentioned $\mathbf{P}$ vectors can be modelled with PH distributions. The Markov process for the $\mathbf{P}^j$ vectors depends on the repair strategy for the bridge in state j.

When no repair is planned for the bridge in state j, the Markov process is a pure deterioration process for $\mathbf{P}^j$ vectors (see Fig. 3).

When minor repairs are planned for the bridge in state 2, the Markov process for $\mathbf{P}^2$ vectors can be modelled as in Fig. 4. Similarly, when major repairs are planned for the bridge in state 3, the Markov process for $\mathbf{P}^3$ vectors are shown in Fig. 5. The Markov process for $\mathbf{P}^4$ and $\mathbf{P}^5$ vectors can be established similarly based on the repair strategies. The detailed process will not be presented in this paper. In the following, the states without superscripts are called the main states ($\mathbb{S}_d$), while the ones labelled with superscripts (e.g. 2.1^1, 2.2^{m_2}) are called states for repair ($\mathbb{S}_r$).

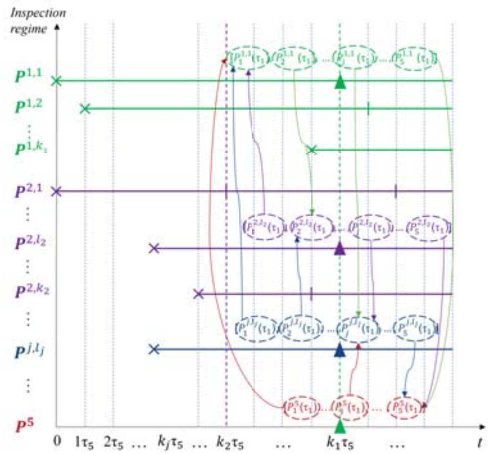

Fig. 2. Modelling of condition-based inspections

To make it consistent in the calculation, we establish transition matrices with the same size. The number of phases for repair m_r takes the maximum value of (m_2, m_3, m_4, m_5), where m_j denotes the number of phases required for approximating the repair delays of state j. The full transition matrix for $\mathbf{P}^j$ vectors can be established as

$$\boldsymbol{A}^j = \begin{pmatrix} \boldsymbol{A}_d & & & & & \\ & \boldsymbol{A}_{2-5} & \boldsymbol{A}^j_{r,1} & & & \\ & & \boldsymbol{A}_{2-5} & \boldsymbol{A}^j_{r,2} & & \\ & & & \ddots & & \\ \boldsymbol{A}^j_{r,m_j} & & & & \boldsymbol{A}_{2-5} & \\ & & & & & \ddots \\ & & & & & & \boldsymbol{A}_{2-5} \end{pmatrix} \quad (5)$$

where $\boldsymbol{A}_d$ contains the transition rates among $\mathbb{S}_d$ (see Eq. 6); $\boldsymbol{A}_{2-5}$ contains the transition rates among macro state 2 to macro state 5 in $\mathbb{S}_d$, which applies for every phase in $\mathbb{S}_r$; $\boldsymbol{A}^j_{r,u}$ contains the transition rates from the uth phase to the $u+1$th phase of $\mathbb{S}_r$ (see Eq. 7). It should be noted that $\boldsymbol{A}^j_{r,m_j}$ represents the transition rates from the last phase of $\mathbb{S}_r$ to $\mathbb{S}_d$, where the success probabilities e_S, e_M and e_L are included based on the repair

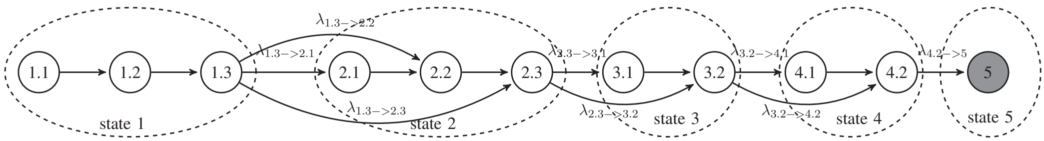

Fig. 3. Markov process for $\mathbf{P}^j$ when there is no repair planned for bridge in state j

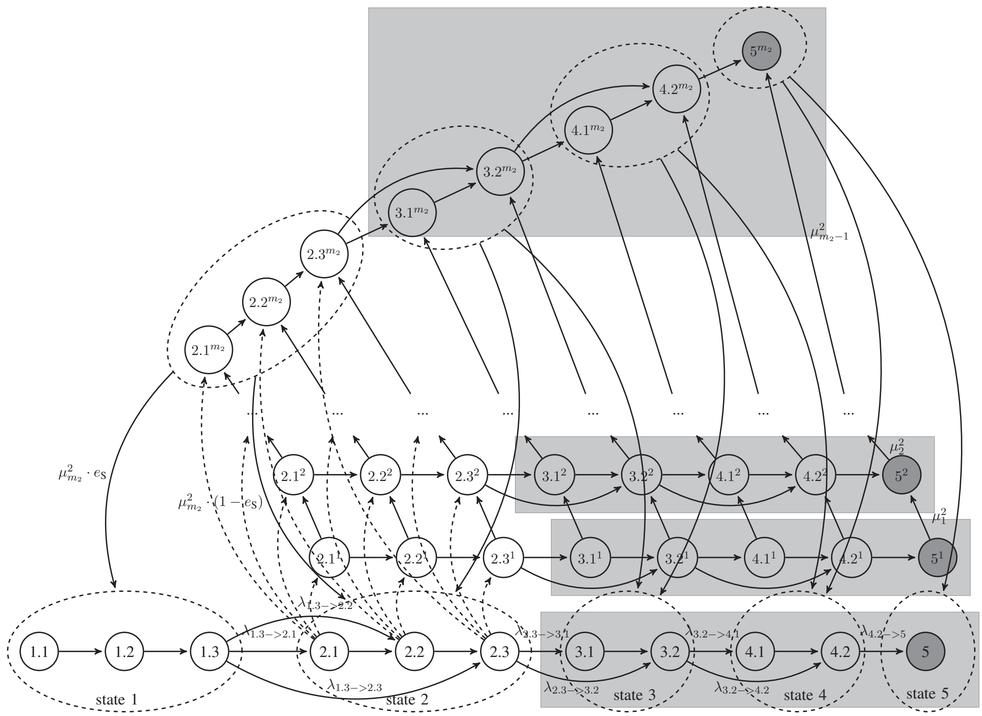

Fig. 4. Markov process for $\mathbf{P}^2$ when minor repairs will be planned for the bridge in state 2

strategy for the bridge in macro state j.

$$\boldsymbol{A}_d = \begin{pmatrix} \boldsymbol{S}_1 & \boldsymbol{S}_{1\to 2} & & & \\ & \boldsymbol{S}_2 & \boldsymbol{S}_{2\to 3} & & \\ & & \boldsymbol{S}_3 & \boldsymbol{S}_{3\to 4} & \\ & & & \boldsymbol{S}_4 & \boldsymbol{S}_{4\to 5} \\ & & & & 0 \end{pmatrix} \tag{6}$$

$$\boldsymbol{A}^j_{r,u} = \text{diag}(\mu^j_u, ..., \mu^j_u) \tag{7}$$

During the inspections, the probability mass in $\mathbb{S}_d$ is moved to $\mathbb{S}_r$, described in Fig. 4 and Fig. 5 as the dashed lines. This process can be realized by multiplying the $\mathbf{P}^j$ vector with its inspection matrices $\boldsymbol{B}^j$. For instance, Eq. 8 gives the inspection matrix for $\mathbf{P}^2$. Here $\boldsymbol{B}^2_u$ is in the dimension of state 2, $\boldsymbol{I}_1$, $\boldsymbol{I}_{3-5}$ and $\boldsymbol{I}_r$ denotes the identity matrix in the dimension of macro state 1, macro state 3 to 5 and the states in $\mathbb{S}_r$ respectively.

Meanwhile, the shuffling of the probability mass described in Fig. 2 will be done, representing a change of inspection regime. It should be noted that here P^j_2 includes the probability mass in state 2.1, 2.2, 2.3, 2.1^1, 2.2^1, 2.3^1, ..., 2.1^{m_r}, 2.2^{m_r}, 2.3^{m_r}; P^j_3 includes the probability mass in state 3.1, 3.2, ..., 3.1^{m_r}, 3.2^{m_r} (see the states marked with the grey box). The same applies to the mod-

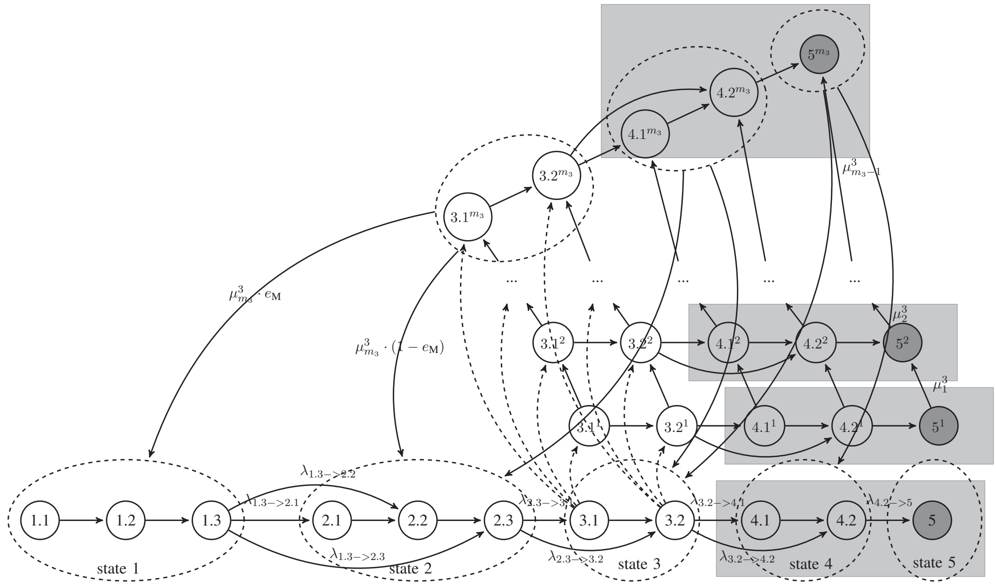

Fig. 5. Markov process for $\mathbf{P}^3$ when major repairs will be planned for the bridge in state 3

elling of P_4^j and P_5^j.

$$\boldsymbol{B}^2 = \begin{pmatrix} \boldsymbol{I}_1 & & & \\ & \boldsymbol{B}_1^2 \cdots \boldsymbol{B}_{m_2}^2 & \cdots & \boldsymbol{B}_{m_r}^2 \\ & & \boldsymbol{I}_{3-5} & \\ & & & \boldsymbol{I}_r \end{pmatrix}, \tag{8}$$

$$\boldsymbol{B}_2^u = \text{diag}(\beta_u^2, ..., \beta_u^2)$$

To ensure assumption (vi), the expected waiting time before a repair should not be extended when shuffling the probability mass. This can be done by updating the inspection matrix. With a reverse summation of the expected sojourn times of the intermediate states, the expected waiting time for the uth phase of $\mathbb{S}_r$ in vector $\mathbf{P}^j$ can be expressed as $\frac{1}{\mu_u^j} + \frac{1}{\mu_{u+1}^j} + ... + \frac{1}{\mu_{m_j}^j}$. Similarly, we can derive the expected waiting times for all $\mathbf{P}$ vectors and move the probability mass to a phase with the closest waiting time.

3.4. *Maintenance optimisation*

A cost function considering the number of actions (inspections and different repairs) is adopted to evaluate the efficiency of a specific strategy.

$$E(C(T)) = \frac{\sum_{t=1}^{T} \sum_{\theta \in \Theta} C_\theta \cdot E(N_\theta(t))}{T} + d_f \cdot C_f \tag{9}$$

where T denotes the total calculation time; Θ denotes the set of possible actions (inspection and different repairs); C_θ denotes the cost per action θ; $E(N_\theta(t))$ denotes the expected number of action θ at year t; d_f denotes the expected duration of being in state 5 and C_f denotes the penalty per unit time when the bridge is in state 5.

The expected number of repairs can be calculated by summating the probability masses from the $\mathbb{S}_r$ back to $\mathbb{S}_d$ at each integration. Based on the intended state improvement, these probability masses are classified into minor repairs, major repairs and rehabilitation.

$$\begin{aligned} E(N_{\text{Rehab}}) &= \sum_{t=0}^{T} \sum_{\phi j > 2} \sum_{i \in \mathbb{S}_d} (P_i^{j,l_j}(t^+) - P_i^{j,l_j}(t^-)) \\ E(N_{\text{MaRep}}) &= \sum_{t=0}^{T} \sum_{\phi j = 2} \sum_{i \in \mathbb{S}_d} (P_i^{j,l_j}(t^+) - P_i^{j,l_j}(t^-)) \\ E(N_{\text{MiRep}}) &= \sum_{t=0}^{T} \sum_{\phi j = 1} \sum_{i \in \mathbb{S}_d} (P_i^{j,l_j}(t^+) - P_i^{j,l_j}(t^-)) \end{aligned} \tag{10}$$

where $P_i^{j,l_j}(t^+)$ denotes the probability mass at state i of vector $\mathbf{P}^{j,l_j}$ right after the integration and $P_i^{j,l_j}(t^+)$ denotes the one right before the integration.

The expected number of inspections can be calculated as a summation of the probability masses in the whole vector at each inspection.

$$E(N_{\text{Insp}}) = \sum_{t=0}^{T} \sum_{t=\tau_{\{j,l_j\}}} \sum_{i \in \mathbb{S}_d \cup \mathbb{S}_r} P_i^{j,l_j}(t) \quad (11)$$

4. Numerical Results

4.1. *Input parameters*

Table 3 presents the bridge deterioration and repair delays parameters. The parameters for bridge deterioration are estimated from a study by Fang and Sun (2019) based on the bridge inspection data in Shanghai. The parameters for repair delays are based on the maintenance requirement in Norway, which can be summarised as follows: For bridges in state 1, no maintenance action is required; for bridges in state 2, maintenance should be conducted between four to ten years; for bridges in state 3, maintenance should be conducted between one to three years; for bridges in state 4 and 5, maintenance should be conducted within six months. Table 4 presents the unit costs for different maintenance actions.

Table 3. Input parameters

State	Deterioration parameters		Repair parameters	
	Scale (yr)	Shape	Mean (yr)	Sigma
1	27.531	1.458	/	/
2	26.025	1.599	4.421	0.142
3	31.788	1.328	3.167	0.149
4	21.266	1.217	1.375	0.149
5	/	/	1.375	0.149

4.2. *Bridge performance and the number of actions*

To verify the proposed model, we compared The result with a Monte Carlo Simulation (MCS), where the sojourn time at each main state follows a Weibull distribution and the times before repairs are lognormal-distributed with the parameters in Table 3.

Consider a maintenance strategy with $\tau_1 = 14$ years, $\tau_2 = 6$ years, $\tau_3 = 2$ years, $\tau_4 = 3$ months and all repairs intend to restore the bridge to state 1, the time-dependent state probabilities from both approaches are presented in Fig. 6, and

Table 4. Cost values

	C_{Insp}	C_{MiRep}	C_{MaRep}	C_{Rehab}
Cost (10^3NOK)	500	1,000	2,000	4,000

the expected number of inspections and repairs are presented in Table 5. As we can see, the proposed approach gives very close results with the MCS regarding the bridge performance and the expected number of actions.

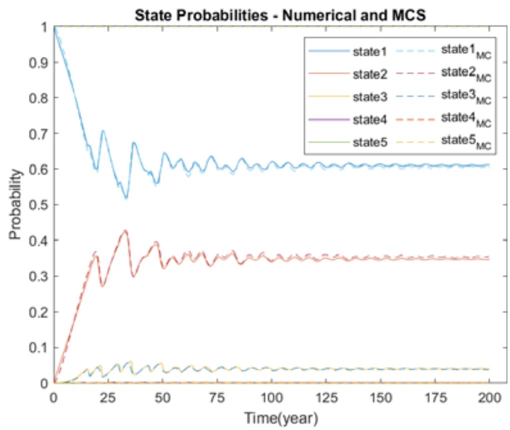

Fig. 6. Time-dependent state probabilities for the illustrative strategy

Table 5. Expected number of actions and system performance

	Proposed Model	MCS
$E(N_{\text{Insp}})$	15.891	15.838
$E(N_{\text{MiRep}})$	3.727	3.689
$E(N_{\text{MaRep}})$	1.184	1.226
$E(N_{\text{Rehab}})$	0.081	0.086
d_f (yr)	0.00091	0.00084

4.3. *Optimisation result*

Different strategies can be evaluated with the proposed approach to search for an optimal solution. In practice, the inspections are usually planned over a time of years. The reference interval τ_5 is set to be one month, and only integer years are considered for τ_1, τ_2 and τ_3. For τ_1, the upper limit for searching is set to be the expected sojourn time while the ones for τ_2 and τ_3 are set to be its expected waiting times before repair. The repair for bridges in state 4 must be done within half a year. Therefore, τ_4 is between 1 month and half a year. Considering a large number of potential solutions, the Genetic Algorithm toolbox in MATLAB is used to search for an optimal solution, with a simulated time of 200 years and a stopping condition of 30 stall generations.

Table 6 presents the optimal solution considering different penalties when the bridge is in state 5. Its expected number of actions, the duration in state 5 and the expected annual costs are evaluated with the MCS of 2,000,000 replications. As we can see, with a higher penalty, the model leads to strategies with more frequent inspections, more early repairs and shorter duration in state 5, which is consistent with our analysis.

Table 6. Optimisation results considering different C_f

C_f (10^3 NOK/yr)	20,000	50,000
Inspection Intervals	$\tau_1 = 12$ years, $\tau_2 = 7$ years, $\tau_3 = 2$ years, $\tau_4 = 6$ months	$\tau_1 = 10$ years, $\tau_2 = 5$ years, $\tau_3 = 2$ years, $\tau_4 = 6$ months
Repair Strategy	μ_2 = MiRep, μ_3 = MaRep, $\mu_4 = \mu_5$ = Rehab	
$E(N_{\text{Insp}})$	16.116	21.48
$E(N_{\text{MiRep}})$	4.269	4.261
$E(N_{\text{MaRep}})$	1.032	1.066
$E(N_{\text{Rehab}})$	0.004	0.004
d_f (yr)	6.193×10^{-4}	2.252×10^{-4}

5. Summary

This paper presents an extension of our earlier work on the PH model considering condition-based inspections and significant delays before the repairs (Sun and Vatn, 2023). In contrast to the deterministic delay times modelled with extra matrices, this paper considered lognormal-distributed delay times and investigated modelling such delays with PH distributions. Monte Carlo Simulation is used to verify the results.

With this model, the expected system performance can be assessed given the inspection intervals, the delays before the repairs and the repair actions at different system conditions. An illustration case of road bridges is presented to demonstrate the modelling approach and its potential use in maintenance optimisation.

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