Proceedings of the 33rd European Safety and Reliability Conference (ESREL 2023)
Edited by Mário P. Brito, Terje Aven, Piero Baraldi, Marko Čepin and Enrico Zio

doi: 10.3850/978-981-18-8071-1_P383-cd

EXACT AND ASYMPTOTIC RESULTS FOR CONNECTED (r,2)-out-of-(m,n):F LATTICE SYSTEMS

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A (r,s)-out-of-(m,n):F system consists in $m \times n$ elements arranged in n rows and m columns; it fails if *all* elements in a block $r \times s$ fail. The interest in such systems has never dwindled since their introduction by Salvia and Lasher (1990), because of the ever increasing number of real-life applications: reliability of electronic devices, X-ray and disease diagnostic, security of communications and property, pattern search systems, etc. Computing their exact availability has been, in the general case, deemed a numerically complex task by Nashwan (2018) and Zhao *et al.* (2011). Only a few configurations have allowed simple solutions.
The special case of (2,2)-out-of-(m,n):F systems has been studied by (Malinowski and Tanguy, 2022), in which exact solutions where provided for $2 \leq m \leq 10$ and arbitrary n through recurrence relations, the order of which increases drastically with m. Based on these results, an analytical, asymptotic expansion was given for large m and n, which was shown to be in excellent agreement for m as low as 4.
In this paper, we generalize our previous work to (r,2)-out-of-(m,n):F systems. We have obtained the exact expressions of the availability for $3 \leq r \leq 8$ and several values of m, while n can remain arbitrary. An asymptotic expansion has again been inferred for arbitrary (large) m and n, which allows quick numerical evaluations. We have also calculated the Mean Time To Failure of such systems, assuming that all elements are identical and obey a Weibull lifetime distribution.

Keywords: Cellular network, connected (r,s)-out-of-(m,n):F lattice system, network reliability, availability, generating function, asymptotic expansion.

1. Introduction and context

A simple description of a telecommunication network may be performed by considering a two-dimensional lattice system, such as those initially proposed by (Salvia and Lasher, 1990; Ksir, 1992; Boehme et al., 1992; Zuo, 1993; Preuss and Boehme, 1994) for a generalization of the well-known k-out-of-n systems. A (r,s)-out-of-(m,n):F system consists in $m \times n$ elements arranged in n rows and m columns; the system fails if *all* elements in a block $r \times s$ fail. An algorithm in O($s^{m-r}\, m^2\, r\, n$) has been published (Yamamoto and Miyakawa, 1995) for the general (r,s)-out-of-(m,n):F lattice system, with a numerical evaluation for $r = 2$, $s = 2$, $m = 4$, and $n = 4, 10, 50$. This effort has attracted a lot of interest from various groups (Khamis and Mokhlis, 1997; Habib et al., 2010; Yamamoto et al., 2008; Zhao et al., 2011; Nashwan, 2015) that developed various improved algorithms, while keeping small values of r, s, and m. A more recent effort (Nakamura et al., 2018) focused on the cases $r = m-1$ and $r = m-2$ for which efficient algorithms were proposed. Other approaches using embedded Markov chains or Monte Carlo computations have been published (Zhao et al., 2009, 2012). More recently (Malinowski, 2021), the case $r = s = 2$ and $2 \leq m \leq 4$ has been revisited, with algorithms of O($n + m$) complexity, calculating reliability through nested recursions.

We cannot give here the credit that they deserve to all the works concerning (r,s)-out-of-(m,n):F configurations, and their many variants. Excellent surveys are found in (Kuo and Zuo, 2003; Akiba et al., 2019). This sustained mathematical effort demonstrates that these configurations have many practical applications, as recognized early by Salvia and Lasher (1990). They identified the reliability of electronic devices as a straightforward application of their calculations. This has remained true (Chang and Mohapatra, 1998; Beiu and Dăuş, 2015; Akiba and Yamamoto, 2001), even though the size of transistors and other devices has shrunk drastically. X-rays and disease diagnostics (Salvia and Lasher, 1990; Hsieh and Chen, 2004) may now be joined by studies of biological systems at the cell scale (Beiu and Dăuş, 2015). Wireless sensor systems for security and communication are now pervading our lives and their reliabilities are of the utmost importance (Makri and Psillakis, 1997; Habib et al., 2010; Cheng et al., 2016; Si et al., 2017; Nakamura et al., 2018; Liu, 2019; Malinowski, 2021). Finally, pattern search systems (Aki and Hirano, 2004; Hsieh and Chen, 2004; Habib et al., 2010) are a crucial topic for the AI techniques that have revolutionized many industrial sectors.

The purpose of this paper is to assess the probability of operation of a (r,2)-out-of-(m,n):F lattice system, $\Pr(B^{r\times 2}_{m\times n})$, extending our previous results (Malinowski and Tanguy, 2022). The general aim is to provide simple, analytical results, that could still give accurate results in essentially O(1) time. We also address the Mean Time To Failure (MTTF) of such systems.

The paper is organized as follows. In Section 2, we start with the case $r = 3$ and $s = 2$, for the smallest possible width ($m = 3$) of interest. We introduce the use of generating functions, which leads to a complete analytical solution of the $m = 3$ case. The quasi-power-law behavior of $\Pr(B^{3\times 2}_{3\times n})$ for large n's is demonstrated when the unavailability of elements, q, is small, because one eigenvalue of the problem prevails over the others. Section 3 deals with the $m = 4$ configuration, still with $r = 3$ and $s = 2$. The methodology used in this work is explained, with an emphasis on the "transfer matrix" approach to the problem. Exact analytical results are provided and already show that as m increases, the solution becomes more complex. Section 4 treats the $m \geq 5$ cases, which have been solved exactly for arbitrary q when $m \leq 12$. It shows that the orders of the recursions increase faster than m, while the quasi-power-law dependence holds. We then repeat in Section 5 all the procedure for $4 \leq r \leq 8$ in order to assess the dependence of the results on r. We devote Section 6 to the assessment of the MTTF of a (r,2)-out-of-(m,n):F lattice system. Our results lead to a general expression in the (r,s)-out-of-(m,n):F configuration. We finally summarize our results and their possible extensions in the Conclusion.

2. Case $\mathrm{r} = 3$, $\mathrm{s} = 2$, and $\mathrm{m} = 3$

The values of $\Pr(B^{3\times 2}_{3\times n}) \equiv R_n$ can be obtained very easily from the recurrence relation

$$R_n = (1 - q^3)\, R_{n-1} + q^3\, (1 - q^3)\, R_{n-2}\,, \quad (1)$$

and the initial conditions $R_0 = 1$ and $R_1 = 1$. These recurrence relations lead to a simple expression using a generating function generally defined by (Stanley, 2011):

$$\mathcal{G}(z) = \sum_{n=0}^{\infty} R_n\, z^n\,. \quad (2)$$

Because of the recurrence relation (1), $\mathcal{G}_3(z)$ is here a rational fraction of z, which reads

$$\mathcal{G}_3(z) = \frac{1 + q^3\, z}{1 - (1 - q^3)\, z - q^3\, (1 - q^3)\, z^2}\,. \quad (3)$$

Its partial fraction decomposition gives

$$\mathcal{G}_3(z) = \frac{\alpha_+}{1 - \zeta_+\, z} + \frac{\alpha_-}{1 - \zeta_-\, z}\,. \quad (4)$$

The power series expansion in z of (4), compared with (2), gives

$$R_n = \alpha_+\, \zeta_+^n + \alpha_-\, \zeta_-^n\,, \quad (5)$$

where

$$\zeta_\pm = \frac{1}{2}\left(1 - q^3 \pm \sqrt{1 + 2\, q^3 - 3\, q^6}\right)\,, \quad (6)$$

$$\alpha_\pm = \frac{1}{2} \pm \frac{1 + q^3}{2\sqrt{1 + 2\, q^3 - 3\, q^6}}\,. \quad (7)$$

The eigenvalues $\zeta_\pm$ are determined by solving the simple quadratic equation deduced from the denominator of (3), while $\alpha_\pm$ are found from solving the system $\{R_0 = 1, R_1 = 1\}$, using (5). Alternatively, $\alpha_\pm$ is the residue associated with the root $1/\zeta_\pm$ of the denominator of $\mathcal{G}_3(z)$. The variations with q of the two eigenvalues are shown in Figure 1. In practice, one is mainly interested in high component availabilities ($q \to 0$). In this regime, $\zeta_+ \to 1$ and $\zeta_- \to 0$, while $\alpha_+ \to 1$ and $\alpha_- \to 0$. When n is large enough, the prevailing term is thus associated with ζ_+:

$$R_n \approx \alpha_+ \, \zeta_+^n \, . \tag{8}$$

This means that for large n's, an accurate estimate of R_n can be obtained. The complexity of the calculation is then O(1), not O(n).

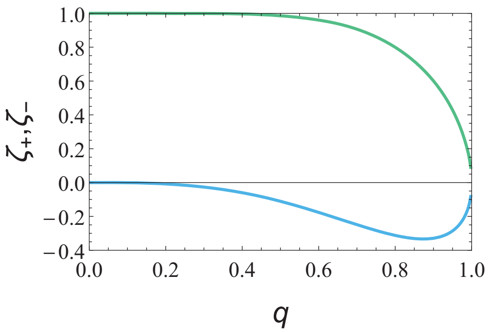

Fig. 1. Variation of the two eigenvalues ζ_+ (green) and ζ_- (blue) when $m = 3$.

3. Detailed derivation for $\mathrm{r} = 3$, $\mathrm{s} = 2$, and $\mathrm{m} = 3$

The aim of this section is to explain the existence of a recurrence relation between successive values of n for $\Pr(B^{3\times 2}_{m\times n})$. The gist of the method is given for $m = 4$ and is readily generalized.

For a lattice of width $m = 4$, the successive objects are replaced by 1 if they are operating, and 0 if they are failed. The state of each row can then be seen as the binary representation of an integer k such that $0 \le k \le 15$ (in the general case, the bounds will be 0 and $2^m - 1$). The system will fail if there exists a 3×2 block of 0's when a new layer (row) is added. For such a configuration to occur, only seven "transitions" are possible, as shown below (note the red 3×2 blocks of zeros).

$$0 \longrightarrow 0 \quad \boxed{\begin{matrix}0&0&0&0\\0&0&0&0\end{matrix}}$$

$$0 \longrightarrow 1 \quad \boxed{\begin{matrix}0&0&0&0\\0&0&0&1\end{matrix}}$$

$$0 \longrightarrow 8 \quad \boxed{\begin{matrix}0&0&0&0\\1&0&0&0\end{matrix}}$$

$$1 \longrightarrow 0 \quad \boxed{\begin{matrix}0&0&0&1\\0&0&0&0\end{matrix}}$$

$$1 \longrightarrow 1 \quad \boxed{\begin{matrix}0&0&0&1\\0&0&0&1\end{matrix}}$$

$$8 \longrightarrow 0 \quad \boxed{\begin{matrix}1&0&0&0\\0&0&0&0\end{matrix}}$$

$$8 \longrightarrow 8 \quad \boxed{\begin{matrix}1&0&0&0\\1&0&0&0\end{matrix}}$$

Consequently, one can merge the states described by the integers 2, 3, 5, 6, and 7 in a single state E (for "Else"). A new set of four states $\mathcal{I} = \{0, 1, 8, \mathrm{E}\}$ must now be considered. Let us denote by $p_i^{(n)}$ the probability that the n-layer lattice is still operating, provided that the last (nth) layer is described by the ith state, with $i \in \mathcal{I}$. We have therefore $p_0^{(n)} = \Pr_n(0)\, p_E^{(n-1)}$ (the only possibility here), where $\Pr_n(0)$ is the probability of occurrence of state 0 in layer n. It is not difficult to consider all the cases and derive

$$\begin{aligned}
p_0^{(n)} &= \Pr_n(0)\, p_E^{(n-1)} \\
p_1^{(n)} &= \Pr_n(1)\, \left(p_8^{(n-1)} + p_E^{(n-1)}\right) \\
p_8^{(n)} &= \Pr_n(8)\, \left(p_1^{(n-1)} + p_E^{(n-1)}\right) \\
p_E^{(n)} &= \Pr_n(E)\, \Big(p_0^{(n-1)} + p_1^{(n-1)} \\
&\qquad + p_8^{(n-1)} + p_E^{(n-1)}\Big)
\end{aligned} \tag{9}$$

For the sake of simplicity, we assume that $\Pr_n(i)$ does not depend on n; its simplified notation will be $p(i)$ from now on. Equation (9) is rewritten as

$$\begin{pmatrix} p_0^{(n)} \\ p_1^{(n)} \\ p_8^{(n)} \\ p_E^{(n)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & p(0) \\ 0 & 0 & p(1) & p(1) \\ 0 & p(8) & 0 & p(8) \\ p(E) & p(E) & p(E) & p(E) \end{pmatrix} \begin{pmatrix} p_0^{(n-1)} \\ p_1^{(n-1)} \\ p_8^{(n-1)} \\ p_E^{(n-1)} \end{pmatrix} \tag{10}$$

Let us denote by M the 4×4 transfer matrix in (10). The calculation of $\Pr(B_{4\times n}^{3\times 2}) = \sum_{i\in\{0,1,8,E\}} p_i^{(n)}$ is easily done since $p_i^{(1)} = p(i)$:

$$\Pr(B_{4\times n}^{3\times 2}) = (1\ 1\ 1\ 1) \cdot M^n \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} . \tag{11}$$

The probability of failure of each element is given by q, so that $p(0) = q^4$, $p(1) = p(8) = q^3\,(1-q)$, and $p(E) = 1-p(0)-p(1)-p(8) = 1-2\,q^3+q^4$. Using the simplified notation $\Pr(B_{4\times n}^{3\times 2}) \equiv R_n$, one must have $R_0 = R_1 = 1$. The next values are

$$\begin{aligned} R_2 &= 1 - 2\,q^6 + q^8 , \\ R_3 &= 1 - 4\,q^6 + 2\,q^8 + 2\,q^9 + 2\,q^{10} - 4\,q^{11} \\ &\quad + q^{12} , \\ R_4 &= 1 - 6\,q^6 + 3\,q^8 + 4\,q^9 + 4\,q^{10} - 8\,q^{11} \\ &\quad + 4\,q^{12} - 4\,q^{13} + 2\,q^{14} , \\ R_5 &= -8\,q^6 + 4\,q^8 + 6\,q^9 + 6\,q^{10} - 12\,q^{11} \\ &\quad + 11\,q^{12} - 8\,q^{13} - 6\,q^{15} - q^{16} + 14\,q^{17} \\ &\quad - 6\,q^{18} - 2\,q^{19} + q^{20} - 2\,q^{11} + 15\,q^{12} \\ &\quad - 6\,q^{13} - 2\,q^{14} + q^{15} . \end{aligned}$$

Because the R_n's rely on the nth power of matrix M, they must obey a recurrence relation the order of which is at most the size of the matrix. In the case $m = 4$, the characteristic polynomial of M is

$$\begin{aligned} \mathcal{P}_{m=4}(X) &= (X + q^3 - q^4) \\ &\times \Big(X^3 - (1-q^3)\,X^2 - q^3\,(1-2\,q^3+q^4)\,X \\ &\quad + q^7\,(1-q)^2\,(1+q+q^2-q^3) \Big) . \end{aligned} \tag{12}$$

The recurrence relation is actually related to the polynomial of degree 3 in X in (12):

$$\begin{aligned} R_n &= (1-q^3)\,R_{n-1} \\ &+ q^3\,(1-2\,q^3+q^4)\,R_{n-2} \\ &- q^7\,(1-q)^2\,(1+q+q^2-q^3)\,R_{n-3} , \end{aligned} \tag{13}$$

with $R_0 = R_1 = 1$, and $R_2 = 1 - 2\,q^6 + q^8$. The first R_n's lead to the associated generating function $\mathcal{G}_4(z) = \mathcal{N}_4(z)/\mathcal{D}_4(z)$, with

$$\mathcal{N}_4(z) = 1 + q^3\,z - (1-q)\,q^7\,z^2 \tag{14}$$

$$\begin{aligned} \mathcal{D}_4(z) &= 1 - (1-q^3)\,z \\ &\quad - q^3\,(1-2\,q^3+q^4)\,z^2 \\ &\quad + q^7\,(1-q)^2\,(1+q+q^2-q^3)\,z^3 \end{aligned} \tag{15}$$

The variations of the three ζ_k with q are displayed in Figure 2. They are such that, again, when n is moderately large and q close to 0, R_n essentially obeys a power law with respect to n:

$$R_n \approx \alpha_+^{(4)} \left(\zeta_+^{(4)}\right)^n . \tag{16}$$

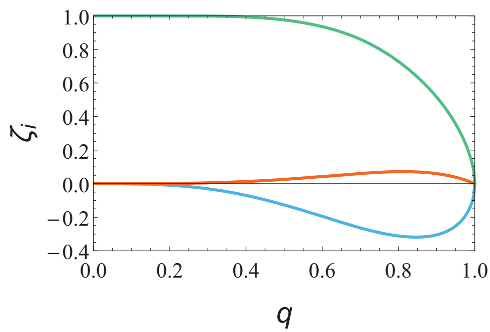

Fig. 2. Variation with q of the three eigenvalues when $m = 4$. $\zeta_+^{(4)}$ is represented by the green curve.

Again, the calculation of the probabilities can be performed in O(1) time.

4. Case m $\geq$ 5 for $r = 3$ and $s = 2$

The previous methodology is also used for $5 \leq m \leq 12$, and the results are similar. The number of eigenvalues, which are all real, increases with m as in the (2,2)-out-of-(m,n) case (Malinowski and Tanguy, 2022). For $3 \leq m \leq 12$, we have respectively 2, 3, 4, 5, 6, 12, 19, 27, 41, and 61 real eigenvalues. The asymptotic behavior of $R_n^{(m)}$ still obeys a power-law expression

$$R_n^{(m)} \approx \alpha_+^{(m)} \left(\zeta_+^{(m)}\right)^n . \tag{17}$$

Taking the logarithm of (17) provides

$$\ln R_n^{(m)} \approx \ln \alpha_+^{(m)} + n \, \ln \zeta_+^{(m)} . \tag{18}$$

Expanding $\ln R_n^{(m)}$ in the limit $q \to 0$ for successive and large enough values of n gives access to the Taylor expansion of $\ln \alpha_+^m$ and $\ln \zeta_+^{(m)}$. This operation can be repeated for various values of m. The variation with m of the results indicates that

$$\zeta_+^{(m)} \to \chi_* \, \zeta_*^m , \tag{19}$$

$$\alpha_+^{(m)} \to \delta_* \, \gamma_*^m , \tag{20}$$

leading to the final asymptotic approximation

$$R_n^{(m)} \approx \delta_* \, (\gamma_*)^m \, (\chi_*)^n \, (\zeta_*)^{m\,n} . \tag{21}$$

We have found (the expansion of the logarithms are kept for use in Section 6, for the assessment of the Mean Time To Failure)

$$\begin{aligned}
\ln \zeta_* &= -q^6 + q^8 + q^9 + 2\,q^{10} - 2\,q^{11} \\
&\quad - \frac{25}{2}\,q^{12} - 6\,q^{13} + 21\,q^{14} + 35\,q^{15} \\
&\quad + \frac{109}{2}\,q^{16} - 60\,q^{17} - \frac{1949}{6}\,q^{18} \\
&\quad - 258\,q^{19} + 510\,q^{20} + \cdots \\
\ln \chi_* &= 2\,q^6 - 3\,q^8 - 2\,q^9 - 6\,q^{10} + 4\,q^{11} \\
&\quad + 44\,q^{12} + 24\,q^{13} - 82\,q^{14} - 126\,q^{15} \\
&\quad - \frac{447}{2}\,q^{16} + 174\,q^{17} + \frac{4379}{3}\,q^{18} \\
&\quad + 1310\,q^{19} - 2333\,q^{20} + \cdots \\
\ln \gamma_* &= q^6 - q^8 - 2\,q^9 - 4\,q^{10} + 4\,q^{11} \\
&\quad + \frac{45}{2}\,q^{12} + 16\,q^{13} - 35\,q^{14} - 90\,q^{15} \\
&\quad - \frac{315}{2}\,q^{16} + 144\,q^{17} + \frac{2440}{3}\,q^{18} \\
&\quad + 842\,q^{19} - 1047\,q^{20} + \cdots \\
\ln \delta_* &= -2\,q^6 + 3\,q^8 + 4\,q^9 + 12\,q^{10} \\
&\quad - 8\,q^{11} - 77\,q^{12} - 60\,q^{13} + 128\,q^{14} \\
&\quad + 316\,q^{15} + \frac{1293}{2}\,q^{16} - 404\,q^{17} \\
&\quad - \frac{10628}{3}\,q^{18} - 4052\,q^{19} + 4360\,q^{20} + \cdots
\end{aligned}$$

Equation (21) has been checked with known exact values obtained in the preceding Sections. The agreement is quite satisfactory for small q's and even moderately large values of m and n. With respect to our previous study Malinowski and Tanguy (2022), we observe that $\ln \chi_*$ and $\ln \gamma_*$ are not identical anymore. The reason is that the 3×2 structure is not symmetrical when we interchange m and n. One expects the behavior given in (21) to hold for patterns that are not symmetric; otherwise one would have $\chi_* = \gamma_*$ as in the $r = s = 2$ configuration. In three-dimensional (r,s,t)-out-of-(m,n,l):F systems, (21) is likely to generalize as a multi-power-law expression, with a $(\zeta_*)^{m\,n\,l}$ prevailing term.

5. Results for various values of *r*

5.1. *Methodology*

In the preceding Section, we obtained a simple asymptotic expression (21) for the availability/reliability of a (3,2)-out-of-(m,n):F system. Each value of ζ_*, χ_*, γ_*, and δ_* depends explicitly on the unavailability q. These parameters also depend *implicitly* on the specific values $r = 3$ and $s = 2$. Our aim in this section is to make some progress in the knowledge of the dependence on r of these expansions.

We have therefore performed the same calculations and processing of the results for $r = 4$ and $s = 2$, with m going from 4 to 13. The degrees of the recurrences were successively 2, 3, 4, 5, 6, 9, 13, 20, 28, 39. Unsurprisingly, the general behavior of (21) was found again, however with different expressions for ζ_*, χ_*, γ_*, and δ_*.

We proceeded similarly for $r = 5$ and $s = 2$, for $5 \le m \le 14$, with recurrences of order 2, 3, 4, 5, 6, 7, 10, 14, 21, 29. For $r = 6$ and $s = 2$, while $6 \le m \le 15$, the successive orders are 2, 3, 4, 5, 6, 7, 8, 11, 15, 22. We observed for the cases $r = 7$ and $r = 8$ the same behaviors, in which the orders of the recurrences increase less rapidly than for lesser values of r. The compilation of all the resulting expansions of $\ln \zeta_*$, etc. allowed us to derive the expressions given in the next subsection.

5.2. *Expansions of ζ_*, χ_*, γ_*, and δ_* as functions of r, for $s = 2$*

After a few polynomial interpolations, one gets

$$\begin{aligned}\ln \zeta_* &= -q^{2r}\,(1-q^2) \\ &+ q^{3\,r}\,(1+2\,q-2\,q^2-q^3) \\ &+ q^{4r}\left[-\frac{6\,r+5}{2} - 6\,q + (6\,r+3)\,q^2 \right. \\ &\qquad \left. + 6\,q^3 - \frac{6\,r+1}{2}\,q^4\right] + \cdots \end{aligned} \tag{22}$$

$$\begin{aligned}\ln \chi_* &= q^{2r}\,[(r-1) - r\,q^2] \\ &+ q^{3r}\,\left[-(r-1) - 2\,r\,q + 2\,(r-1)\,q^2 + r\,q^3\right] \\ &+ q^{4r}\left[\frac{9\,r^2+2\,r-5}{2} + 8\,r\,q \right. \\ &\qquad - (9\,r^2+3\,r-8)\,q^2 - (8\,r-4)\,q^3 \\ &\qquad \left. + \frac{9\,r^2+4\,r}{2}\,q^4\right] + \cdots \end{aligned} \tag{23}$$

$$\begin{aligned}\ln \gamma_* &= q^{2r}\,(1-q^2) \\ &- 2\,q^{3\,r}\,(1+2\,q-2\,q^2-q^3) \\ &+ q^{4r}\left[\frac{10\,r+11}{2} + 16\,q - (10\,r+5)\,q^2 \right. \\ &\qquad \left. - 16\,q^3 + \frac{10\,r-1}{2}\,q^4\right] + \cdots \end{aligned} \tag{24}$$

$$\begin{aligned}\ln \delta_* &= -q^{2r}\,[(r-1) - r\,q^2] \\ &- 2\,q^{3r}\,\left[-(r-1) - 2\,r\,q + 2\,(r-1)\,q^2 + r\,q^3\right] \\ &- q^{4r}\left[\frac{15\,r^2+6\,r-11}{2} + 20\,r\,q \right. \\ &\qquad - (15\,r^2+5\,r-22)\,q^2 - (20\,r-12)\,q^3 \\ &\qquad \left. + \frac{15\,r^2+4\,r}{2}\,q^4\right] + \cdots \end{aligned} \tag{25}$$

These expressions could be useful for evaluating the availability not only when m and n are very large, but even when they are moderately so, because q is small in most practical cases. Going back to the $r = s = 2$ case (Malinowski and Tanguy, 2022), one should get $\chi_* = \gamma_*$. This is not obvious at first sight from the above equations, but it actually works, because the decompositions in powers of q^r mix things for these two quantities.

Keeping the prevailing term when $q \to 0$ gives

$$\begin{aligned}&R_n^{(m)}(r, s=2) \to \\ &\quad \exp\left(-(n-1)\,(m-r+1)\,q^{2r}\right) \\ &\quad \times (1 + \text{smaller terms in O}(q))\end{aligned} \tag{26}$$

6. Application of the results: the Mean Time To Failure

In the preceding Section, we have obtained an analytical expression for the asymptotic reliability of a (r,2)-out-of-(m,n):F system. One could also use the exact results for a better definition of upper and lower bounds, following the method of (Malinowski, 2021). From the exact expression of the reliability or availability (depending on the context), we could also address the total system failure rate ν for repairable systems (Yuge et al., 2000), using the formula valid for identical elements with a failure rate λ

$$\nu = \lambda\,p\,\frac{dR_n^{(m)}}{dp} = -\lambda\,(1-q)\,\frac{dR_n^{(m)}}{dq} \tag{27}$$

In this Section, we consider a key performance index of the system, namely the Mean Time To Failure (MTTF) and assume that all equipments' lifetime distributions obey a Weibull law with a form factor β, that is $q(t) = 1 - \exp\left(-(\lambda\,t)^\beta\right)$. The usual formula

$$\text{MTTF} = \int_0^\infty R(t)\,dt \tag{28}$$

can be rewritten after a change of variable as

$$\begin{aligned}\text{MTTF} = \frac{1}{\lambda}\int_0^1 \frac{R_n^{(m)}(q)}{1-q} \\ \times \frac{1}{\beta}\,\left[-\ln(1-q)\right]^{\frac{1}{\beta}-1}\,dq\,.\end{aligned} \tag{29}$$

When n and m are large, one can replace $R_n^{(m)}(q)$ by the expression in (26). Only in the region $q \to 0$ does the integrand have a meaningful contribution, and the prevailing term to be summed is essentially

$$\frac{1}{\beta\,\lambda}\,q^{\frac{1}{\beta}-1}\,e^{-(n-1)\,(m-r+1)\,q^{2r}}\,.$$

Using $X = (n-1)\,(m-r+1)\,q^{2r}$ as a new variable (the upper bound can then be safely replaced by $+\infty$) allows the determination of the asymptotic MTTF in (r,2)-out-of-(m,n):F systems

$$\mathrm{MTTF}(r, s = 2) \to \frac{1}{\lambda} \frac{\Gamma\left(1 + \frac{1}{\beta\, 2\, r}\right)}{[(n-1)\,(m-r+1)]^{\frac{1}{\beta\, 2\, r}}} \quad (30)$$

While this result is the prevailing term in the asymptotic expansion of the MTTF, one should recall that there are extra O(q) terms in the integrand, leading to corrections with a n and m dependence that vanish very slowly. When $m = r - 1$, the MTTF is expected to be infinite, so that the formula exhibits the correct behavior.

Because of the form of (30), and the symmetry of the problem when swapping m and n, as well as r and s, it is natural to expect that in the general (r,s)-out-of-(m,n):F case, one should obtain

$$\mathrm{MTTF}(r, s) \to \frac{1}{\lambda} \frac{\Gamma\left(1 + \frac{1}{\beta\, r\, s}\right)}{[(n-s+1)\,(m-r+1)]^{\frac{1}{\beta\, r\, s}}} \quad (31)$$

The decrease of the MTTF with m and n is very slow. Preliminary calculations for $3 \leq s \leq 5$ confirm (31). This expression can be easily generalized for three-dimensional (or larger) systems.

7. Conclusion and outlook

We have proposed a derivation of the exact recurrence relations of the availability of (r,2)-out-of-(m,n):F lattice systems, thereby extending the results of our previous endeavor concerned with $r = 2$. The obtained results could provide helpful upper and lower bounds of configurations with large values of m, as proposed by (Malinowski, 2021). Our asymptotic, power-law expression (21) can give accurate values with a minimum numerical effort, even for repairable systems. We have determined the prevailing term in the asymptotic expansion of the MTTF, in agreement with numerical values even when m and n are not very large.

This work can be extended in several directions. Firstly, we have already begun to consider larger values of s: 3, 4, 5. It appears that in the case $s = 3$, the eigenvalues are not all real anymore, as they are in the $s = 2$ case. The dependence of the logarithms of ζ_*, χ_*, γ_*, and δ_* with q and r is not as simple as in (22)–(25) and requires further study. Our procedure can also be used for "circular" two-dimensional systems. Finally, it would be useful to have a better picture of $\zeta_*(q)$ for non-vanishingly small values of the unavailability q.

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