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A New Method for Reliability Evaluation of Two-Terminal Multistate Networks in Terms of *d*-minimal cuts

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Many real-world complex systems can be modelled by multistate networks. Theoretically, the evaluation of multistate network reliability is NP-hard. Therefore, it is essential to develop more efficient methods to analyse the reliability of practical multistate networks. There are mainly direct and indirect methods for evaluating reliability. In this paper, we focus on the third stage of the indirect method which is calculating the union probability of the events given all *d*-minimal cuts (*d*-MCs). Based on the reliability evaluation method proposed by Provan and Ball (1984) on binary networks, this study attempt to develop its extended version for the multistate network scenario. The correctness and effectiveness of the proposed method is verified by illustrative example and several benchmark networks.

Keywords: Reliability, Multistate network, Two-terminal network, *d*-MC, MC.

1. Introduction

Many complex real-world systems can be viewed as networks, which facilitates the analysis of their essential characteristics. Network reliability theory has become a popular tool for evaluating the performance of complex systems, such as communication networks (Frank and Hakimi, 1965), transportation networks (Doulliez and Jamoulle, 1972), power transmission and distribution networks (Ke and Wang, 1997), oil/gas production and transportation networks (Aven, 1987), and unmanned swarm networks (Xu, 2022). In binary networks, the components (nodes or links) are assumed to exist in only two states - success or failure. Components in multistate networks can be in multiple states due to performance degradation, allowing for more accurate characterization of real systems. This study focuses on evaluating the reliability of a two-terminal multistate network, which contains a source node and a sink node. The components in network may take independent and identically distributed discrete non-negative integer values. Two-terminal multistate network is suitable for the modelling of realistic complex systems where there is a provider and a demander and transmission links of material, energy or information flows between them. For example, in a power transmission and distribution network, each multistate component represents a transmission line, the source and sink nodes represent the power plant and consumer, respectively, and the demand is the amount of power required by each consumer. The reliability of two-terminal multistate network has been defined in a great deal of research (Jane et al., 1993; Satitsatian and Kapur, 2006; Zuo et al., 2007; Bai et al., 2015; Niu et al., 2017; Xu et al., 2022) as the probability that the required demand flow (d+1 units in this study) can be successfully sent from a source to a sink through the multistate components.

The evaluation of the reliability of two-terminal multistate networks is an NP-hard problem, as shown in theoretical study (Ball, 1986). Currently, the evaluation of multistate network reliability is mainly divided into direct and indirect methods, with the main difference between them being whether the minimal cut vectors/ minimal path vectors satisfying the level *d* demands (*d*-MCs or *d*-MPs for short). In this study, since the indirect method draws on the idea of binary network reliability evaluation based on the minimal cut set (path set), and reduces the complexity of the algorithm by solving the problem in stages, we focus on the indirect method. *d*-MCs or *d*-MPs are also known as the upper and lower boundary points of the network demand level *d* (Hudson and Kapur, 1983a). As the definitions of cut and path are dual, this paper focuses on cut-based methods.

The cut-based indirect method is generally divided into three main stages, i.e., 1) finding all minimal cuts (MCs) in the network; 2) enumerating all *d*-MCs based on the given MCs; and 3) computing the union probability of the given *d*-MCs to obtain the reliability value. All stages of the indirect method are NP-hard. For the first stage, a large number of reports have been proposed to solve the MCs enumeration problem (Tsukiyama et al., 1980; Mishra and Chaturvedi, 2009). Subsequently, based on all MCs of the network, Jane et al. (1993) pioneered a mathematical planning model for generating *d*-MCs by MCs. Most of the existing methods are improvements of Jane's method (Yeh et al., 2015; Niu et al., 2017). For the third stage, network reliability is evaluated by all *d*-MCs. The first methods based on inclusion-exclusion (IE) principle, including Hudson and Kapur (1983b) and more recently improved by Hao et al. (2019). The second is the sum of disjoint product (SDP) principle, where Zuo et al. (2007) proposed a recursive method called the recursive sum of disjoint product (RSDP), and Bai et al. (2015) proposed several heuristics to improve the efficiency of RSDP. The third method is the indirect version of state space decomposition (SSD) by Doulliez and Jamoulle (1972), where Aven (1987) proposed an SSD method based on *d*-MC*s* or *d*-MCs to solve multistate network reliability. Then, Bai et al. (2018) improved this method to make it more efficient. Subsequently, they used a parallel decomposition mechanism with heuristics to further improve the efficiency (Bai et al., 2020). In addition to these three popular methods, the binary decision diagram method (Kuo et al., 1999; Yeh et al., 2002; Pan et al., 2022) and the multistate multivalued decision diagram method (Shrestha et al., 2010) can also be used in the third stage.

Then, the question becomes if there is an alternative method in the third stage to indirectly calculate the multistate network reliability. After literature research, we discovered a recursive algorithm proposed in Provan and Ball (1984) for calculating the reliability of binary networks based on MCs. We believe it can be extended to multistate scenarios. In this work, we propose a new method to calculate the multistate network reliability based on *d*-MCs. The rest of this paper is organized as follows. Section 2 describes briefly the algorithm proposed by Provan and Ball. In Section 3, we develop a multistate network version of the algorithm. The effectiveness of the proposed algorithm is verified in Section 4.

2. Provan and Ball's algorithm

Let $G = (N, E)$ denote an undirected or directed network, where $N = \{v_1, \dots, v_n\}$ is the set of nodes and $E = \{e_1, \dots, e_m\}$ is the set of links. s and t denote the source and sink nodes of the network, respectively. Therefore, the binary network reliability measure is defined by the existence of an operating path between a node pair in G. That is, for a specified source-sink node pair s and t the following events are defined.

$$EP(s,t) = [there\ exists\ an\ operating\ path\ from\ s\ to\ t].$$

The minimal cut in the network is denoted by C. Suppose the number of cut in the network is μ. Let C_i denote the i_{th} MC in the network, $i = 1,2,\dots,\mu$. The following set is defined for C_i:

$$SN(C_i) = \{u \mid there\ exists\ a\ path\ from\ source\ s\ to\ u\ containig\ no\ links\ of\ C_i\}.$$

$$TN(C_i) = \{v \mid there\ exists\ a\ path\ from\ node\ v\ to\ sink\ t\ containig\ no\ links\ of\ C_i\}.$$

The network is divided into two sub-networks after removing all the links in C_i. The source node s and the sink node t are blocked, $SN(C_i)$ and $TN(C_i)$ are the sets of nodes containing s and t, which are disjoint from each other, and C_i is the set of links connecting $SN(C_i)$ and $TN(C_i)$. Subsequently, Provan and Ball gave the definition of the ***exit nodes*** corresponding to C_i as follows.

$$SE(C_i) = \{u \mid u \in SN(C_i)\ and\ there\ exists\ a\ link\ (u,v)\ with\ v \in TN(C_i)\}.$$

Then, two essential auxiliary events are given based on several definitions above.

(i) $E(C_i) = [all\ links\ in\ C_i\ fail]$.
(ii) $EC(C_i) = EP\big(s, SE(C_i)\big) \cap E(C_i) = [there\ is\ an\ operating\ path\ from\ s\ to\ all\ \boldsymbol{exit\ nodes}\ of\ C_i, and\ all\ links\ in\ C_i\ fail]$.

In the bridge network shown in Fig. 1 (Jane et al., 1993), the MCs consists of $C_1 = \{e_1, e_5\}$, $C_2 = \{e_1, e_3, e_6\}$, $C_3 = \{e_2, e_4, e_5\}$, and $C_4 = \{e_2, e_6\}$. For C_3, $SN(C_3) = \{s, 1\}$, $TN(C_3) = \{2, t\}$, and $SE(C_3) = \{s, 1\}$. Further, $EC(C_3)$ represents the event that the operating path from s to $SE(C_3)$ ($\{s, 1\}$) exists and the links e_2, e_4, e_5 in C_3 all fail.

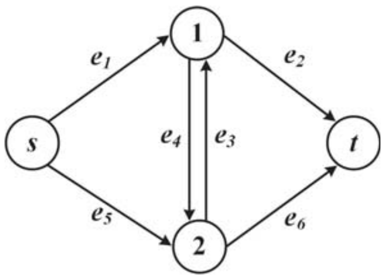

Fig. 1. A bridge network.

To calculate the reliability $Pr[EP(s,t)]$ of a two-terminal binary network based on all MCs in the given network, i.e., $C_1, C_2, \dots, C_\mu$, Provan and Ball (1984) gave the following theorem.

Theorem 1.

$$Pr[EP(s,t)] = 1 - Pr\left[\overline{EP(s,t)}\right] = 1 - \sum_{i=1}^{\mu} Pr[EC(C_i)]. \quad (1)$$

In the process of obtaining the network reliability by Theorem 1, we need to calculate $Pr[EC(C_i)]$. Thus, they provide and prove a recursive formula, which is Theorem 2 below.

Theorem 2. For any C_i,

$$Pr[EC(C_i)] = \prod_{e \in C_i} q_e \left\{1 - \sum_{j=1}^{\mu_i'} \frac{Pr\left[EC(C_j')\right]}{\prod_{e \in \left(C_j' \cap C_i\right)} q_e}\right\}, \quad (2)$$

where q_e denotes the failure probability of component e. μ_i' indicates the number of MCs that precede C_i, which are all C_j' that satisfy $SN(C_j') \subset SN(C_i)$. If $C_j' \cap C_i = \emptyset$, the denominator in parentheses is 1.

Based on Theorems 1 and 2, Provan and Ball provide a recursive algorithm to compute the reliability of a two-terminal binary network, and the procedure as follows.

Step 1: Enumerate all MCs in network G, i.e., $C_1, C_2, \dots, C_\mu$, and ordered in increasing cardinality of $SN(C_i)$.

Step 2: Calculate $Pr[EC(C_i)]$ for each of the ordered MCs using Eq. (2).

Step 3: Evaluate the reliability of the two-terminal binary network, $Pr[EP(s,t)]$, with Eq. (1).

3. The proposed algorithm

Based on Provan and Ball's algorithm for binary network reliability, we attempt to extend it to multistate scenarios.

3.1. *Preliminary*

In a multistate network, the component e_i can work in different states, we denote a state vector of the multistate network by $\boldsymbol{x} = (x_1, x_2, \dots, x_m)$, and x_i means the operational state of the component e_i. Each state takes a discrete stochastic integer value, and the maximum capacity of e_i is W_i, $0 \le x_i \le W_i$. The state distribution describes the operating probability of each component in different states. For example, the state distribution of components in Fig. 1 from Jane et al. (1993) is shown in Table 1.

Table 1. State distributions of the components in the example network.

Component \ State	0	1	2	3
e_1	0.05	0.10	0.25	0.60
e_2	0.10	0.30	0.60	0
e_3	0.10	0.90	0	0
e_4	0.10	0.90	0	0
e_5	0.10	0.90	0	0
e_6	0.05	0.25	0.70	0

The third step of the indirect method to evaluate the multistate network reliability is based on the given d-MCs. The state vector $\boldsymbol{x}$ is d-MC if $\varphi(\boldsymbol{x}) \le d$ and $\varphi(\boldsymbol{y}) > d$ for any $\boldsymbol{y} > \boldsymbol{x}$, where $\varphi(\bullet)$ is a system structure function that represents the mapping relationship from component state vectors to network states. $\boldsymbol{y} > \boldsymbol{x}$ implies that for every component there is $y_i \ge x_i$, but there exists at least one component e_j satisfying $y_j > x_j$, $i, j = 1, \dots, m$.

For calculating multistate network reliability, i.e., the probability of sending at least $d+1$ units of demand flows successfully from s to t, the general methods based on IE principle, SDP principle or SSD principle use all L d-MCs ($\boldsymbol{z}^1, \boldsymbol{z}^2, \dots, \boldsymbol{z}^L$) given to calculate their union probability to obtain the reliability as follows.

$$Pr[\varphi(\boldsymbol{x}) \ge d+1] = 1 - Pr[\varphi(\boldsymbol{x}) \le d] = 1 - Pr[\{\boldsymbol{x} \le \boldsymbol{z}^1\} \cup \{\boldsymbol{x} \le \boldsymbol{z}^2\} \cup \cdots \cup \{\boldsymbol{x} \le \boldsymbol{z}^L\}]. \quad (3)$$

Inspired by Provan and Ball's algorithm, instead of using Eq. (3) directly to compute the union probability of d-MCs to obtain the reliability, we extend the algorithm to multistate scenarios by applying the computational idea and framework described by Provan and Ball.

3.2. *Definition, property and theorem*

In the proposed new algorithm, the key problem is to extend the object of Provan and Ball's algorithm from MC to d-MC. First, we attempt to extend the events to the multistate case.

(i) In binary networks, the reliability is defined as the occurrence of the event $EP(s,t)$. For multistate networks, we define the reliability measure by the events as follows.

$$FP(s,t,d+1) = [the\ flow\ from\ s\ to\ t\ is\ greater\ than\ or\ equal\ to\ demand\ d+1].$$

(ii) In binary networks, given a MC C_i, the event that all the links in C_i fail is indicated by $E(C_i)$. For multistate networks, according to the maximum-flow minimum-cut theorem (Fulkerson and Ford, 1962), for C_i so that the sum of the flows of all links through it is less than or equal to d, it is unable to transport $d+1$ units of flows from s to t. Therefore, we define the following events.

$$F(C_i,d) = [the\ sum\ of\ flows\ of\ all\ links\ in\ C_i\ is\ less\ than\ or\ equal\ to\ the\ demand\ d].$$

(iii) In binary networks, $EC(C_i)$ indicates the event that there are operating paths from s to all ***exit nodes*** of C_i and all links in C_i fail. To get its multistate version, we discuss $EC(C_i)$ as follows.

The network is separated by each MC into two sub-networks, one containing s and the other containing t. For C_i, let $N_s(C_i)$ and $N_t(C_i)$ denote the nodes set including s and t, respectively. It is obvious that $SN(C_i) \subset G_s(C_i)$, $TN(C_i) \subset G_t(C_i)$ and $N_s(C_i) \cap N_t(C_i) = \emptyset$. For the nodes set X and Y, define $E(X,Y) = \{(x,y) \in E | x \in X, y \in Y\}$, meaning all links between nodes in X and nodes in Y. We associate a new network, $G'(C_i)$ with each C_i, in which $G'(C_i) = N_s(C_i) \cup E(N_s(C_i), N_s(C_i)) \cup C_i \cup t'$, where t' is a new node virtualized by node set $N_t(C_i)$. For instance, consider $C_3 = \{e_2, e_4, e_5\}$ in the bridge network of Fig. 1, and the new network $G'(C_3)$ is shown in Fig. 2.

There are two MCs in the obtained new network in Fig. 2, i.e., $C_1 = \{e_1, e_5\}$ and $C_3 = \{e_2, e_4, e_5\}$. For C_i, the definition of its 'Previous' MCs (in short **P-MCs**) is given as:

$$P\text{-}MC(C_i) = \{C_j' \mid the\ minimum\ cut\ C_j'\ contained\ in\ G'(C_i)\ except\ C_i\ and\ satisfying\ SN(C_j') \subset SN(C_i)\}.$$

where the number of P-MCs of C_i is denoted by μ_i'. In this example, $P\text{-}MC(C_3) = \{C_1\}$, satisfying $SN(C_1) \subset SN(C_3)$. Hence, we define the event $FC(C_i,d)$ as follows.

$$FC(C_i,d) = \left(\bigcap_{j=1}^{\mu_i'} \overline{F(C_j',d)}\right) \cap F(C_i,d) = [the\ sum\ of\ flows\ of\ all\ links\ in\ each\ \boldsymbol{P\text{-}MCs}\ of\ C_i\ is\ greater\ than\ demand\ d,\ but\ the\ summation\ of\ flows\ of\ all\ links\ of\ C_i\ is\ small\ than\ or\ equal\ to\ d].$$

In summary, incorporating the above events and the theorems in binary networks, we develop the corresponding theorems for multistate networks.

Theorem 3.

$$Pr[FP(s,t,d+1)] = 1 - Pr[\overline{FP(s,t,d+1)}] = 1 - \sum_{i=1}^{\mu} Pr[FC(C_i,d)]. \quad (4)$$

Eq. (4) is the equation for multistate network reliability, where $\overline{FP(s,t,d+1)}$ is the complementary event. Importantly and not negligibly, the μ MCs are sorted in increasing order of the cardinality of $SN(C_i)$.

In the evaluation of reliability with Theorem 3, it is necessary to calculate the probability of the event $FC(C_i,d)$. Therefore, we propose the following recursive formula.

Theorem 4.

$$Pr[FC(C_i,d)] = Pr[F(C_i,d)] - \sum_{j=1}^{\mu_i'} Pr\left[FC(C_j',d) \cap F(C_i,d)\right]. \quad (5)$$

In the following we will discuss the calculation of $Pr[F(C_i,d)]$ and $Pr\left[FC\left(C_j',d\right) \cap F(C_i,d)\right]$.

Firstly, most existing algorithms are based on the mathematical model proposed by Jane et al. (1993) to generate all d-MCs. The model determines that each real d-MC is generated by a particular MC. We denote all sets of real d-MCs generated from C_i as $\boldsymbol{Z}_i$, whose number is L_i. Then we denote the jth of the d-MCs by $\mathbf{z}_i^j$, i.e., $\boldsymbol{Z}_i = (\mathbf{z}_i^1, \mathbf{z}_i^2, \dots, \mathbf{z}_i^{L_i})$. For example, $\mathbf{z}_\mu^{L_\mu}$ means the last $L_\mu th$ real d-MC generated by the last MC C_μ in the network. Since d-MC is a special state vector of the network, one of the d-MCs can be expressed as $\mathbf{z}_i^j = (z_i^j(1), z_i^j(2), \dots, z_i^j(m))$. Thereby we give the following properties:

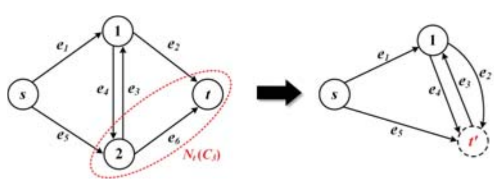

Fig. 2. A new network after virtualization of a node set.

Property 1.

$$Pr[F(C_i,d)] = Pr[\{\boldsymbol{x} \le \boldsymbol{z}_i^1\} \cup \{\boldsymbol{x} \le \boldsymbol{z}_i^2\} \cup \cdots \cup \{\boldsymbol{x} \le \boldsymbol{z}_i^{L_i}\}] = Pr[\{\boldsymbol{x} \le \boldsymbol{Z}_i\}]. \quad (6)$$

where $\boldsymbol{x}$ is the state vector of the multistate network. $\boldsymbol{x} \le \boldsymbol{Z}_i$ denotes an event where the state vector is less than or equal to at least one *d*-MC. Thus, we associate the event $F(C_i,d)$ defined with *d*-MC. If given all the real *d*-MCs $\boldsymbol{Z}_i$ generated by C_i, the union probability of all $\boldsymbol{z}_i^j$ can be calculated by the RSDP algorithm proposed by Zuo et al. (2007), yielding the probability of the event $F(C_i,d)$.

Then, to compute $Pr[FC(C_j',d) \cap F(C_i,d)]$, based on $FC(C,d)$ and P-MCs, combined with set theory and the basic additive law of probability (Zuo et al., 2007), we give the following properties:

Property 2.

$$Pr[FC(C_j',d) \cap F(C_i,d)] = Pr[F(C_j',d) \cap F(C_i,d)] - \sum_{k=1}^{\mu_j''}\{(-1)^{k-1}\sum_{\substack{H\subset\{1,2,\ldots,\mu_j''\}\\|H|=k}} Pr[(\cap_{h\in H} F(C_h'',d)) \cap F(C_j',d) \cap F(C_i,d)]\}. \quad (7)$$

where C_h'' is the *hth* MC in $P\text{-}MC(C_j')$ and μ_j'' denotes the number of them. Obviously, if C_j' is C_1, then $Pr[FC(C_1,d) \cap F(C_i,d)] = Pr[F(C_1,d) \cap F(C_i,d)]$.

The intersection probability of the events $FC(C,d)$ and $F(C,d)$ can be converted into the probability of only the event $F(C,d)$ by Property 2. We first introduce a special "minimization" intersection operator, "$\oplus$", from Zuo et al. (2007), which is defined for two *d*-MCs $\boldsymbol{z}_j^1$ and $\boldsymbol{z}_i^1$ as:

$$\boldsymbol{z}_j^1 \oplus \boldsymbol{z}_i^1 = \left(min\left(\boldsymbol{z}_j^1(k), \boldsymbol{z}_i^1(k)\right)\right), 1 \le k \le m. \quad (8)$$

$\boldsymbol{z}_{j,i}^t$ denotes the t *th* state vector generated by the symbolic operation of the two *d*-MCs belonging to $\boldsymbol{Z}_j$ and $\boldsymbol{Z}_i$, respectively, which are combined as $\boldsymbol{Z}_{j,i}$, i.e., $\boldsymbol{Z}_{j,i} = (\boldsymbol{z}_{j,i}^1, \boldsymbol{z}_{j,i}^2, \ldots, \boldsymbol{z}_{j,i}^{L_i*L_j})$. Therefore, define $\boldsymbol{x} \le \boldsymbol{Z}_{j,i}$ as the event whose state vector is less than or equal to at least one $\boldsymbol{z}_{j,i}^t$. Suppose that to calculate $Pr[F(C_j,d) \cap F(C_i,d)]$, the following property are obtained:

Property 3.

$$Pr[F(C_j,d) \cap F(C_i,d)] = Pr\left[\{\boldsymbol{x} \le \boldsymbol{z}_{j,i}^1\} \cup \{\boldsymbol{x} \le \boldsymbol{z}_{j,i}^2\} \cup \cdots \cup \{\boldsymbol{x} \le \boldsymbol{z}_{j,i}^{L_j*L_i}\}\right] = Pr[\{\boldsymbol{x} \le \boldsymbol{Z}_{j,i}\}]. \quad (9)$$

Therefore, we can use the RSDP algorithm to calculate the union probability of all $\boldsymbol{z}_{j,i}^t$ to obtain $Pr[F(C_j',d) \cap F(C_i,d)]$ in Eq. (7). In addition, the intersection probability of multiple events, the second half of Eq. (7), can be evaluated by the associative law of event operations.

Based on the properties above, we can compute $Pr[FC(C_i,d)]$ according to the ordered MCs, and by using Eq. (4) to evaluate the multistate network reliability.

3.3. *The algorithm*

The algorithm for evaluating the reliability of multistate networks for given all MCs and *d*-MCs according to theorems 3 and 4 is provided below.

Input: A multistate network $G(N,E)$, a source node s and a sink node t, and all MCs $C_1, C_2, \ldots, C_\mu$ as well as *d*-MCs $\boldsymbol{z}^1, \boldsymbol{z}^2, \ldots, \boldsymbol{z}^L$ and the state distribution of the components.

Output: The reliability of multistate networks $Pr[FP(s,t,d+1)]$.

Step 1: Set the unreliability to $U = 0$. Sort all MCs $C_1, C_2, \ldots, C_\mu$ in increasing order of the cardinality of $SN(C_i)$. Meanwhile, all *d*-MCs $\boldsymbol{z}^1, \boldsymbol{z}^2, \ldots, \boldsymbol{z}^L$ are grouped so that the *d*-MCs generated by the same C_i are divided into one group as $\boldsymbol{Z}_i$, so it can be divided into μ groups: $\boldsymbol{Z}_1, \boldsymbol{Z}_2, \ldots, \boldsymbol{Z}_\mu$, and the number of *d*-MCs in $\boldsymbol{Z}_i$ $(1 \le i \le \mu)$ is L_i.

Step 2: For each C_i, a new network is obtained by virtualizing the set of nodes $N_t(C_i)$ to obtain $P\text{-}MC(C_i)$ with the number of μ_i'.

Step 3: Let $i = 1$.

Step 4: Calculate $Pr[FC(C_i,d)]$ according to the following steps.

Step 4.1: If the group $\boldsymbol{Z}_i$ of *d*-MCs divided by C_i is empty, then go to **Step 5**. Otherwise, for $\boldsymbol{Z}_i$, compute the union probability $Pr[\{\boldsymbol{x} \le \boldsymbol{z}_i^1\} \cup \{\boldsymbol{x} \le \boldsymbol{z}_i^2\} \cup \cdots \cup \{\boldsymbol{x} \le \boldsymbol{z}_i^{L_i}\}]$ using the RSDP algorithm, thus $Pr[F(C_i,d)]$ is obtained.

Step 4.2: If $i = 1$, then $Pr[FC(C_i,d)] = Pr[F(C_i,d)]$ and go to **Step 4.8**.

Step 4.3: Set $j = 1$，$U' = 0$.

Step 4.4: For the jth P-MC C'_j of C_i. If $\boldsymbol{Z}_{j'} = \emptyset$, turn to **Step 4.6**. Otherwise, the RSDP method is used to calculate the union probability of $\boldsymbol{Z}_{j',i} = (\boldsymbol{z}^1_{j',i}, \boldsymbol{z}^2_{j',i}, \dots, \boldsymbol{z}^{L_j*L_i}_{j',i})$ by combining Eq. (8) and (9) to obtain the intersection probability $Pr\left[F(C'_j, d) \cap F(C_i, d)\right]$. Then, the intersection probability $Pr\left[(\bigcap_{h\in H} F(C''_h, d)) \cap F(C'_j, d) \cap F(C_i, d)\right]$ of multiple events is calculated in the same way. Finally, $\left[FC(C'_j, d) \cap F(C_i, d)\right]$ can be calculated according to Eq. (7).

Step 4.5: Let $U' \leftarrow U' + Pr\left[FC(C'_j, d) \cap F(C_i, d)\right]$.

Step 4.6: If $j < \mu'_i$, let $j = j + 1$, and return to **Step 4.4**.

Step 4.7: Set $Pr[FC(C_i, d)] = Pr[F(C_i, d)] - U'$.

Step 4.8: Let $U \leftarrow U + Pr[FC(C_i, d)]$.

Step 5: If $i < \mu$，let $i = i + 1$, go to **Step 4**.

Step 6: Calculate the multistate network reliability $Pr[FP(s, t, d + 1)] = 1 - U$ and halt.

4. Numerical experiments

In this section, we select three benchmark networks and corresponding component state distributions to develop the correctness and effectiveness verification experiments of the proposed algorithm, including the illustrated network (as in Fig. 3(1)) and distribution network (as in Fig. 3(3)) given by Niu et al. (2017a) and Niu et al. (2017b), respectively, and the example network (as in Fig. 3(2)) in Satitsatian and Kapur (2006). The maximum capacity vectors of components in the above three networks are: $\boldsymbol{W}_1 = (4, 3, 1, 4, 3, 3)$, $\boldsymbol{W}_2 = (3, 2, 2, 3, 1, 2, 1, 2, 3, 4)$ and $\boldsymbol{W}_3 = (3, 5, 3, 3, 3, 3, 4, 2, 4, 2, 4, 5, 4)$, respectively. The proposed algorithm is coded in MATLAB and experiments are performed on a PC with Windows 10, AMD R7-5800H @ 3.20 GHz CPU and 32 GB RAM.

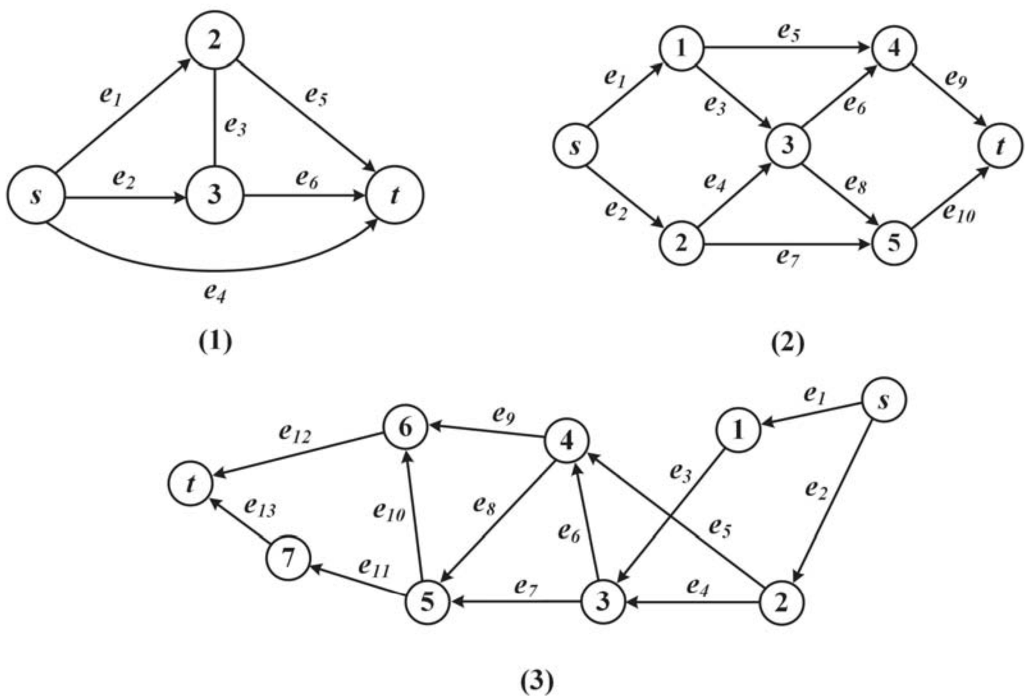

Fig. 3. Three multistate benchmark networks.

Table 2. Experimental results.

Network	n	m	μ	D	d	L	$Pr[FP(s,t,d+1)]$	$Pr[FP(s,t,d+1)] - Pr[FP(s,t,d+2)]$
1	4	6	4	10	0	4	0.9999979609	0.0000110243
					1	14	0.9999869366	0.0000366214
					2	30	0.9999503152	0.0003644755
					3	46	0.9995858397	0.0011633368
					4	57	0.9984225029	0.0031910695
					5	59	0.9952314334	0.0142564324
					6	50	0.9809750010	0.0410279776
					7	34	0.9399470234	0.0796847169
					8	18	0.8602623065	0.1128785915
					9	7	0.7473837150	0.7473837150
2	7	10	16	5	0	16	0.9584005464	0.2292835068
					1	44	0.7291170396	0.3670531164
					2	50	0.3620639232	0.2646256452
					3	36	0.0974382780	0.0887452860
					4	17	0.0086929920	0.0086929920
3	9	13	20	8	0	20	0.9998068112	0.0008096615
					1	64	0.9989971497	0.0021656313
					2	140	0.9968315184	0.0109543662
					3	203	0.9858771522	0.0230548355
					4	204	0.9628223167	0.0607254285
					5	140	0.9020968882	0.1240965164
					6	73	0.7780003718	0.1687363188
					7	34	0.6092640530	0.6092640530

The experimental results of the algorithm under the three benchmark networks are illustrated in Table 2. Where n and m denote the number of nodes and edges of the network, respectively. μ indicates the number of all cut sets in the network. The maximum flow of the network at the maximum capacity vector is denoted by D. For more comprehensive verification of the algorithm, different demand levels d can be selected for each network. The number of all d-MCs generated by the cut sets is represented by L. $Pr[FP(s,t,d+1)]$ denotes the probability of ability to transmit the required d+1 demand flows from the source node s to the sink node t of the multistate network, i.e., the network reliability.

The reliability $Pr[FP(s,t,d+1)]$ of the three multistate networks in Table 2 at different demand levels are the same as the results of Niu et al. (2017a, 2017b) and Satitsatian and Kapur (2006). Therefore, the correctness and effectiveness of the proposed algorithm is proved. Meanwhile, the results in the table demonstrate that the number L of d-MCs in each network initially increases and then decreases as the demand level d rises. The reason is that in the process of generating d-MCs based on MC by using the mathematical planning model in Jane et al. (1993), if the combination of states of all components in MC need to satisfy a larger or smaller value of d, the fewer such combinations (i.e., the number of d-MCs denoted as L) are generated. Conversely, as d approaches the middle range, more d-MCs are generated, similar to the binomial coefficient. Furthermore, the reliability is decreasing as the demand level d increases, because the larger the demand, the less network component state vectors are available to satisfy the condition, and the accumulative probability value of reliability is decreasing. In addition, we calculate the difference $Pr[FP(s,t,d+1)] - Pr[FP(s,t,d+2)]$ between the reliabilities corresponding to adjacent demand levels. This difference represents the probability that accurate d+1 flows can be successfully transmitted from source s to sink t.

5. Conclusions

In this study, we discussed the binary network reliability algorithm proposed by Provan and Ball, which was successfully extended to the multistate network scenario. The corresponding essential definitions, properties, and theorems are given, a new indirect method based on *d*-MCs for calculating the reliability of multistate networks is proposed. Referring to this method it is possible to calculate the reliability accurately for a given network with all *d*-MCs. In addition, the computational procedure of the algorithm is illustrated and the correctness and effectiveness of the proposed algorithm is verified on several benchmark networks.

In the future, we attempt to extend the algorithm to larger-scale networks to find modifications that significantly improve the efficiency of the algorithm. The efficiency comparison analysis with previous accurate algorithms for multistate network reliability will be conducted. Meanwhile, the application of multistate network reliability evaluation algorithm to realistic complex systems will improve computational efficiency and provide managers and designers with assistance in system maintenance and optimization. For example, for the equipment support transportation network with military significance, the multistate network is constructed with the component states of transportation ability/road capacity of different routes selected by the support unit, and the intervention of the solution algorithm can provide the commanders with decision-making assistance in identifying the most reliable equipment support transportation routes to achieve rapid support and accelerate the operation response time.

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References

Aven, T. (1985). Reliability evaluation of multistate systems with multistate components. *IEEE Transactions on reliability* 34 (5), 473-479.

Aven, T. (1987). Availability evaluation of oil/gas production and transportation systems. *Reliability engineering* 18 (1), 35-44.

Bai, G. H., M. J. Zuo, and Z. G. Tian (2015). Ordering heuristics for reliability evaluation of multistate networks. *IEEE Transactions on Reliability* 64 (3), 1015-1023.

Bai, G. H., Z. G. Tian, and M. J. Zuo (2018). Reliability evaluation of multistate networks: An improved algorithm using state-space decomposition and experimental comparison. *IISE Transactions* 50 (5), 407-418.

Bai, G. H., T. Liu, Y. A. Zhang, and J. Y. Tao (2020). An improved method for reliability evaluation of two-terminal multistate networks based on state space decomposition. *IEEE Transactions on reliability* 70 (3), 1084-1095.

Ball, M. O. (1986). Computational complexity of network reliability analysis: An overview. *IEEE transactions on reliability* 35 (3), 230-239.

Doulliez, P. and E. Jamoulle (1972). Transportation networks with random arc capacities. *Revue française d'automatique, informatique, recherche opérationnelle. Recherche opérationnelle* 6 (V3), 45-59.

Ford Jr, L. R. and D. R. Fulkerson (1962). *Flows in networks.* Princeton, Princeton University Press.

Frank, H. and S. Hakimi (1965). Probabilistic flows through a communication network. *IEEE Transactions on Circuit Theory* 12 (3), 413-414.

Hao, Z. F., W. C. Yeh, J. Wang, G. G. Wang, and B. Sun (2019). A quick inclusion-exclusion technique. *Information Sciences* 486, 20-30.

Hudson, J. C. and K. C. Kapur (1983a). Modules in coherent multistate systems. *IEEE transactions on reliability* 32 (2), 183-185.

Hudson, J. C. and K. C. Kapur (1983b). Reliability analysis for multistate systems with multistate components. *IIE Transactions* 15 (2), 127-135.

Jane, C. C., J. S. Lin, and J. Yuan (1993). Reliability evaluation of a limited-flow network in terms of minimal cutsets. *IEEE transactions on reliability* 42 (3), 354-361.

Ke, W. J. and S. D. Wang (1997). Reliability evaluation for distributed computing networks with imperfect nodes. *IEEE Transactions on Reliability* 46 (3), 342-349.

Kuo, S. Y., S. K. Lu, and F. M. Yeh (1999). Determining terminal-pair reliability based on edge expansion diagrams using OBDD. *IEEE Transactions on Reliability* 48 (3), 234-246.

Mishra, R. and S. K. Chaturvedi (2009). A cutsets-based unified framework to evaluate network reliability measures. *IEEE Transactions on reliability* 58 (4), 658-666.

Niu, Y. F., Z. Y. Gao, and W. H. Lam (2017a). A new efficient algorithm for finding all d-minimal cuts in multi-state networks. *Reliability Engineering & System Safety* 166, 151-163.

Niu, Y. F., Z. Y. Gao, and W. H. Lam (2017b). Evaluating the reliability of a stochastic distribution network in terms of minimal cuts. *Transportation Research Part E: Logistics and Transportation Review* 100, 75-97.

Pan, Z. S., W. B. Li, and F. L. Lin (2022). A new heuristic ranking method for BDD network reliability analysis. *Journal of Zhejiang Normal University* (*Natural Science Edition*) 04, 378-387.

Provan, J. S. and M. O. Ball (1984). Computing network reliability in time polynomial in the number of cuts. *Operations research* 32 (3), 516-526.

Satitsatian, S. and K. C. Kapur (2006). An algorithm for lower reliability bounds of multistate two-terminal networks. *IEEE Transactions on Reliability* 55 (2), 199-206.

Shrestha, A., L. D. Xing, and D. W. Coit (2010). An efficient multistate multivalued decision diagram-based approach for multistate system sensitivity analysis. *IEEE Transactions on Reliability* 59 (3), 581-592.

Tsukiyama, S., I. Shirakawa, H. Ozaki, and H. Ariyoshi (1980). An algorithm to enumerate all cutsets of a graph in linear time per cutset. *Journal of the ACM* (*JACM*) 27 (4), 619-632.

Xu, B., T. Liu, G. H. Bai, J. Y. Tao, Y. A. Zhang, and Y. N. Fang (2022). A multistate network approach for reliability evaluation of unmanned swarms by considering information exchange capacity. *Reliability Engineering & System Safety* 219, 108221.

Yeh, F. M., S. K. Lu, and S. Y. Kuo (2002). OBDD-based evaluation of k-terminal network reliability. *IEEE Transactions on Reliability* 51 (4), 443-451.

Yeh, W. C., C. Bae, and C. L. Huang (2015). A new cut-based algorithm for the multi-state flow network reliability problem. *Reliability Engineering & System Safety* 136, 1-7.

Zuo, M. J., Z. G. Tian, and H. Z. Huang (2007). An efficient method for reliability evaluation of multistate networks given all minimal path vectors. *IIE transactions* 39 (8), 811-817.